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# A FATOU THEOREM FOR THE SOLUTION OF THE HEAT EQUATION AT THE CORNER POINTS OF A CYLINDER

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ABSTRACT. In this paper the author proves existence and uniqueness of the initial-Dirichlet problem for the heat equation in a cylindrical domain $D \times (0, \infty)$ where $D$ is a bounded smooth domain in $R^n$ with zero lateral values. A unique representation of the strong solution is given in terms of measures $\mu$ on $D$ and $\lambda$ on $\partial D$. We also show that the strong solution $u(x, t)$ of the heat equation in a cylinder converges a.e. $x_0 \in \partial D \times \{0\}$ as $(x, t)$ converges to points on $\partial D \times \{0\}$ along certain nontangential paths.

## INTRODUCTION

The existence and uniqueness of the initial-Dirichlet problem for the heat equation in a cylindrical domain $D \times (0, T)$ subject to Dirichlet boundary conditions $u|_{\partial D \times (0, T)} \equiv 0$ where $D$ is a bounded smooth domain in $R^n$ have been studied by a large number of researchers. (See [F, LSU, FGS].)

In this paper, by following the argument of Dahlberg and Kenig [DK2], I prove the existence and uniqueness of the nonnegative strong solution of the initial Dirichlet problem (IDP) for the heat equation in a cylinder $D \times (0, \infty)$, $D \in C^\infty$, with Dirichlet boundary condition $u_{\partial D \times (0, \infty)} \equiv 0$.

In fact I show that corresponding to each nonnegative strong solution $u(x, t)$ of IDP, there exists a pair of measures $\mu$ on $D$ and $\lambda$ on $\partial D$ such that

$$\begin{aligned} u(x, t) = & \int_D G(X, t; Q, 0)\, d\mu(Q) \\ & + \int_{\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)\, d\lambda(Q) \end{aligned}$$

where $G(x, t; Q, s)$ is the Green function for the heat equation and $\partial/\partial N_Q$ is the derivative in the direction of the inward normal at $Q$.

I also find that the strong solution $u(x, t)$ of the heat equation in a cylinder converges a.e. $x_0 \in \partial D \times \{0\}$ as $(x, t)$ converges to points on $\partial D \times \{0\}$ along certain nontangential path. In fact I prove that

$$\lim_{\substack{(x, t) \in \overline{\Gamma}_\beta(x_0) \\ t \to 0}} u(x, t) = \frac{\beta}{\sqrt{4\pi}} \cdot \frac{d\lambda}{d\sigma} \quad \text{a.e. } x_0 \in \partial D \times \{0\}$$

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where $\overline{\Gamma}_\beta(x_0) = \{(x', x_n, t) \in D \times (0, T) : |x'| \le Cx_n, \ x_n = \beta t^{3/2}\}$, $x = (x', x_n)$ is the local coordinate of the point $x$ with respect to the local coordinate system at $x_0$ with origin at $x_0$ and with the plane $\{x_n = 0\}$ tangent to $\partial D$ at $x_0$, and $\frac{d\lambda}{d\sigma}$ is the Radon-Nikodym derivative of $d\lambda$ with respect to the surface measure $d\sigma$ on $\partial D$.

While the existence of Fatou type limit on the lateral surface and bottom of a cylinder has been investigated by Fabes, Garofalo, and Salsa [FGS], Fabes and Salsa [FS] in the case of nondegenerate parabolic equation, and by Kemper [K] in the case of heat equation, nothing is known about the behaviour of the solution $u(x, t)$ as $(x, t)$ tends to $\partial D \times \{0\}$. My result is *entirely new*. I have also shown that the index $\frac{3}{2}$ appearing in the definition of $\overline{\Gamma}_\beta(x_0)$ is *essentially sharp*.

By using the argument of M. Grüter and K. O. Widman [GW], we will establish various estimates for the Green's function of the heat equation in $D \times (0, \infty)$ in §1. (The author was informed by Professor Russell Brown that similar estimates were obtained by E. B. Davies [D] using logarithmic Sobolev inequalities.) In subsection 2.1, we will show that any strong solution $u$ of the (IDP) has a trace $\mu$ on $D$ and a trace $\lambda$ on $\partial D$ with $\int_{x \in D} \delta(x)\, d\mu(x) < \infty$, $\int_{Q \in \partial D} d\lambda(Q) < \infty$, $\delta(x) = \operatorname{dist}(x, \partial D)$ and $\forall \eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$,

$$\lim_{t \to 0} \int_D u(x, t)\eta(x)\, dx = \int_D \eta\, d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N}\, d\lambda$$

following the same line of proof as [DK2].

In subsection 2.3, we will prove some priori estimates for the strong solutions $u$ of the (IDP) of the heat equation in a cylinder. In subsection 2.4, we will use the methods in [JK], [K] and [FGS] to prove the convergence a.e. on $\partial D \times \{0\}$ of $u(x, t)$ as $(x, t) \to (Q_0, 0) \in \partial D \times \{0\}$ along the nontangential paths $\Gamma_\beta(Q_0)$ by assuming the everywhere convergence a.e. of the solution $\overline{u}$ of the heat equation at the corner points $\partial D \times \{0\}$ with initial trace $(d\mu, d\lambda) = (0, d\sigma)$ where $d\sigma$ is the surface measure on $\partial D$ along these nontangential paths.

And in subsection 2.5, we will finish the proof by proving the everywhere convergence of such solution $\overline{u}$ along

$$\begin{aligned}\overline{\Gamma}_\beta(Q_0) = \{(Q, s) = (Q', Q_n, s) \in R^{n-1} \times R^+ \times R^- :& \\ |Q'| \le MQ_n, \ Q_n = \beta(-s)^{3/2} \le \alpha\}&\end{aligned}$$

for all corner points $(Q_0, 0) \in \partial D \times \{0\}$ by using layer potential method. Finally in subsection 2.6 we will show that the index $\frac{3}{2}$ on $t$ in the definition of the nontangential cones at corner points are essentially sharp.

## 1. Estimates for the Green function

In this section we will establish various estimates on $G(x, t; Q, s)$ the Green kernel of the heat equation in $D \times (0, \infty)$ following basically the line of proofs of [GW] for the estimates on the Green's function for the Laplacian. We will start with a lemma.

**Lemma 1.1.** *Let $u_R$ be a solution of the heat equation in $D_R \times (0, (2R)^2)$ where $D_R = B(0, 2R) \backslash \overline{B(0, R)}$ with boundary value given by $\phi(x/2R, t/(2R)^2)$ where*

*$\phi \in C^\infty(\partial_p(D_1 \times (0, \infty)))$ with*

$$\begin{cases} \phi(x, t) \equiv 0 & \text{for } (x, t) \in \partial B(0, 1) \times [0, 1) \cup (B(0, 1)\backslash B(0, 1/2)) \times \{0\}, \\ \phi(x, t) \equiv 0 & \text{for } (x, t) \in \partial B(0, 1/2) \times (1/2, 1), \end{cases}$$

*and $0 \le \phi \le 1$. Then*

$$\|\nabla_x u_R(x, t)\|_{L^\infty(D_R \times (0, (2R)^2))} \le \frac{C}{R} < \infty$$

*for some constant $C > 0$ independent of $R$.*

*Proof.* The lemma follows from the boundary Schauder's estimate. (See [F, p. 65, Theorem 6].)

**Corollary 1.2.** *Let $v_R$ be a solution of the backward heat equation $\partial_s v_R + \Delta v_R = 0$ in $D_R \times (-(2R)^2, 0)$ where $D_R = B(0, 2R)\backslash\overline{B(0, R)}$ with boundary value $\phi$ where $\phi \in C^\infty$ and*

$$\begin{cases} \phi(Q, s) \equiv 1 & \text{for } (Q, s) \in \partial B(0, 2R) \times (-(2R)^2, 0] \\ & \text{or } (Q, s) \in (B(0, 2R)\backslash\overline{B(0, R)}) \times \{0\}, \\ \phi(Q, s) \equiv 0 & \text{for } (Q, s) \in \partial B(0, R) \times (-(2R)^2, -2R^2) \end{cases}$$

*and $0 \le \phi \le 1$. Then*

$$\|\nabla_x v_R(x, t)\|_{L^\infty(D_R \times (-(2R)^2, 0))} \le \frac{C}{R} < \infty$$

*for some constant $C > 0$ independent of $R$.*

**Lemma 1.3.** *For $0 \le s < t \le T$, $T > 0$, we have*

(i)

$$G(P, t; Q, s) \le \frac{C}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)}, \qquad P, Q \in D,$$

(ii)

$$G(P, t; Q, s) \le \frac{C\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)}, \qquad P, Q \in D,$$

(iii)

$$G(P, t; Q, s) \le \frac{C\delta(P)\delta(Q)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)}, \qquad P, Q \in D,$$

(iv)

$$\left|\frac{\partial G}{\partial N_Q}(P, t; Q, s)\right| \le \frac{C\delta(P)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)}, \qquad P \in D, \ Q \in \partial D,$$

*where $\partial/\partial N_Q$ is the derivative in the direction of the normal to $\partial D$ at the point $Q \in \partial D$, $\delta(P) = \operatorname{dist}(P, \partial D)$, and $C$ is a constant independent of $P, Q, t, s$.*

*Proof.* (i) is proved in [LSU]. To prove (ii), note that since $D$ is smooth, $\partial D$ satisfies the exterior sphere condition, i.e. there exists a positive constant $h > 0$ such that for each $x \in \partial D$, $\forall 0 < r \le h$, there exists $x_0 \in D^c$ such that $\overline{B(x_0, r)} \cap \overline{D} = \{x\}$.

*Case* 1. $\delta(P) \ge h$.

Then

$$1 \le \frac{\delta(P)}{h} \le \frac{\delta(P)}{h} \cdot \frac{(t-s)^{1/2}}{(t-s)^{1/2}} \le C \frac{\delta(P)}{(t-s)^{1/2}}.$$

So by (i)

$$\begin{aligned} G(P, t; Q, s) &\le \frac{C'}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \cdot \frac{\delta(P)}{(t-s)^{1/2}} \\ &\le C' \frac{\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)}. \end{aligned}$$

*Case* 2. $\delta(P) \ge (t-s)^{1/2}/4$.

Then $1/4 \le \delta(P)/(t-s)^{1/2}$. So by (i)

$$\begin{aligned} G(P, t; Q, s) &\le \frac{C}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \cdot \frac{4\delta(P)}{(t-s)^{1/2}} \\ &\le C' \frac{\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)}. \end{aligned}$$

*Case* 3. $\delta(P) \le h$ and $\delta(P) \le (t-s)^{1/2}/4 \le h$.

We fix $P^* \in \partial D$ with $|P - P^*| = \delta(P)$ and set $R = (t-s)^{1/2}/4$. Since $D$ satisfies the exterior sphere condition, there exists a $P_0 \in D^c$ such that $\overline{B(P_0, R)} \cap \overline{D} = \{P^*\}$. We may also assume without loss of generality that $\overline{PP^*} \subset D$.

Consider the cylinder $Q'_R = D'_R \times (t-(2R)^2, t]$, $D'_R = B(P_0, 2R) \backslash \overline{B(P_0, R)}$. Then $(P, t) \in \overline{Q'_R} \cap D \times (0, \infty)$ and for any $(P', t') \in \overline{Q'_R} \cap \overline{D} \times (0, \infty)$,

$$\begin{aligned} t - s \ge t' - s &\ge t - (2R)^2 - s \\ &\ge t - s - \left(\frac{(t-s)^{1/2}}{2}\right)^2 = \frac{3(t-s)}{4}. \end{aligned}$$

Now we have either

(a) $$|t-s|^{1/2} \le \frac{|P-Q|}{2}$$

or

(b) $$|t-s|^{1/2} \ge \frac{|P-Q|}{2}.$$

If (a) holds, then $R \le |P-Q|/8$, so

$$\begin{aligned} |P' - Q| &\ge |P - Q| - |P - P'| \\ &\ge |P - Q| - (|P - P_0| + |P_0 - P'|) \\ &\ge |P - Q| - (2R + 2R) \\ &\ge \frac{|P-Q|}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} G(P', t'; Q, s) &\le \frac{C}{(t'-s)^{n/2}} e^{-c|P'-Q|^2/(t'-s)} \quad \text{by (i)} \\ &\le \frac{C'}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)}. \end{aligned}$$

If (b) holds, then $|P-Q|^2/(t-s) \le 4 \Rightarrow e^{-c|P-Q|^2/(t-s)} \ge e^{-4C'}$. Therefore

$$
\begin{aligned}
G(P', t'; Q, s) &\le \frac{C}{(t'-s)^{n/2}} e^{-c|P'-Q|^2/(t'-s)} \quad \text{by (i)} \\
&\le \frac{C}{(t'-s)^{n/2}} \le \frac{C''}{(t-s)^{n/2}} e^{-c'|P-Q|^2/(t-s)}.
\end{aligned}
$$

In both cases, we have

$$
G(P', t'; Q, s) \le C \frac{1}{(t-s)^{n/2}} e^{-c'|P-Q|^2/(t-s)}.
$$

If $\overline{u}_R(P', t') = u_R(P' - P_0, t' - (t - (2R)^2))$, $(P', t') \in \overline{Q'_R}$ where $u_R$ is as in Corollary 5.2, then since

$$
\begin{cases}
G(P', t'; Q, s) = 0 \\
\qquad \overline{u}_R(P', t') \ge 0, \quad \forall (P', t') \in \partial D \times [t - (2R)^2, t) \cap \overline{Q'_R}
\end{cases}
$$

and

$$
\begin{cases}
G(P', t'; Q, s) \le C \dfrac{1}{(t-s)^{n/2}} e^{-c'|P-Q|^2/(t-s)}, \\
\qquad \overline{u}_R(P', t') \equiv 1, \quad \forall (P', t') \in \partial_p Q'_R \cap D \times (0, \infty),
\end{cases}
$$

by applying the maximum principle to the functions

$$
G(\cdot, \cdot; Q, s) \quad \text{and} \quad C \frac{1}{(t-s)^{n/2}} e^{-c'|P-Q|^2/(t-s)} \overline{u}_R(\cdot, \cdot)
$$

in the region $Q_R \cap D \times (0, \infty)$ we have:

$$
G(P, t; Q, s) \le C \frac{1}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \overline{u}_R(P, t)
$$

since $(P, t) \in \overline{Q'_R} \cap D \times (0, \infty)$.

$$
\le C \frac{\delta(P)}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \cdot \left| \frac{\partial \overline{u}_R}{\partial l_{P^*}} (\overline{P}, t) \right|
$$

where $\partial/\partial l_{P^*}$ is the derivative in the direction $\overline{PP^*}$ and $\overline{P}$ is some point on $\overline{PP^*}$.

$$
\begin{aligned}
&\le C \frac{\delta(P)}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \cdot \frac{C'}{(t-s)^{1/2}} \quad \text{(by Corollary 1.2)} \\
&\le C' \frac{\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)}.
\end{aligned}
$$

*Case* 4. $\delta(P) \le h$ and $\delta(P) \le h \le 4h \le (t-s)^{1/2}$.

We fix $P^* \in \partial D$ as before and set $R = h$. Since now $R = h \le (t-s)^{1/2}/4$, the same arguments as in Case 3 imply

$$\begin{aligned} G(P, t; Q, s) &\le C \frac{1}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \overline{u}_R(P, t) \\ &\le C \frac{\delta(P)}{(t-s)^{n/2}} e^{-c|P-Q|^2/(t-s)} \\ &\le C_T \frac{\delta(P)}{(t-s)^{n+1/2}} e^{-c|P-Q|^2/(t-s)} \quad \text{since } 0 \le s < t \le T. \end{aligned}$$

*Proof of* (iii). The proof of (iii) is analogous to the proof of (ii).

(iv) follows from (iii) by dividing both sides of (iii) by $\delta(Q)$ and then letting $Q \to Q_0 \in \partial D$ along the normal to $\partial D$ at $Q_0$.

**Corollary 1.4.** *The following is true*:

(i) $$|\nabla_Q G(P, t; Q, s)| \le C \frac{\delta(P)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)},$$

(ii) $$|\nabla_P G(P, t; Q, s)| \le C \frac{\delta(Q)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)},$$

(iii) $$|\nabla_Q \nabla_P G(P, t; Q, s)| \le C \frac{1}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)},$$

*for all* $0 \le s < t \le T$ *(for any fixed* $T > 0$*)*, $P, Q \in D$, *and* $C$ *is a constant depending only on* $T > 0$.

*Proof. Case* 1. $\delta(Q) \le (t-s)^{1/2}$.

Apply Schauder interior estimate to the function $G(P, t, \cdot, \cdot)$ in the cylinder $Q'_R = B(Q, R) \times (s, s+R^2)$ with $R = \delta(Q)/2$. We get $\forall i = 1, \dots, n$,

$$\begin{aligned} \left| \frac{\partial G}{\partial Q_i}(P, t; Q, s) \right| \cdot \frac{\delta(Q)}{2} &\le C \sup_{(Q', s') \in Q'_R} G(P, t; Q', s') \\ &\le C \sup_{(Q', s') \in Q'_R} \frac{\delta(P)\delta(Q')}{(t-s')^{n+2/2}} e^{-c|P-Q'|^2/(t-s')}. \end{aligned}$$

Now for any $(Q', s') \in Q'_R$,

$$\begin{aligned} & s \le s' \le s + R^2 \\ \Rightarrow\ & t - s \ge t - s' \ge t - s - R^2 \\ & = t - s - \left(\frac{\delta(Q)}{2}\right)^2 \ge t - s - \left(\frac{(t-s)^{1/2}}{2}\right)^2 \\ & \ge \frac{3(t-s)}{4}. \end{aligned}$$

Let $Q^*$ be a point on $\partial D$ such that $|Q - Q^*| = \delta(Q)$. Then

$$\begin{aligned} \delta(Q') &\le |Q' - Q^*| \le |Q' - Q| + |Q - Q^*| \\ &\le \frac{1}{2}\delta(Q) + \delta(Q) = \frac{3\delta(Q)}{2}. \end{aligned}$$

Also we have either

(a) $$(t-s)^{1/2} \le \frac{|P-Q|}{2},$$

or

(b) $$(t-s)^{1/2} > \frac{|P-Q|}{2}.$$

If (a) holds, then

$$\begin{aligned}|P-Q'| &\ge |P-Q| - |Q-Q'| \ge |P-Q| - R \\ &\ge |P-Q| - \left(\frac{\delta(Q)}{2}\right) \ge |P-Q| - \frac{(t-s)^{1/2}}{2} \\ &\ge |P-Q| - \frac{|P-Q|}{4} = \frac{3|P-Q|}{4}.\end{aligned}$$

So

$$e^{-c|P-Q'|^2/(t-s')} \le e^{-c'|P-Q|^2/(t-s)}.$$

If (b) holds, then

$$|P-Q|^2/(t-s) \le 4 \Rightarrow e^{-c'|P-Q|^2/(t-s)} \ge e^{-4C'} \ge e^{-c'|P-Q'|^2/(t-s')}$$

since $e^{-c'|P-Q'|^2/(t-s')} \le 1$. In both cases we have

$$e^{-c'|P-Q'|^2/(t-s')} \le Ce^{-c'|P-Q|^2/(t-s)}.$$

Combining the above inequalities we have

$$\sup_{(Q',s')\in Q'_R} \frac{\delta(P)\delta(Q')}{(t-s')^{n+2/2}} e^{-c|P-Q'|^2/(t-s')} \le C\frac{\delta(P)\delta(Q)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)}.$$

Hence

$$\begin{aligned}\left|\frac{\partial G}{\partial Q_i}(P,t;Q,s)\right| \cdot \frac{\delta(Q)}{2} &\le C\frac{\delta(P)\delta(Q)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)} \\ \Rightarrow |\nabla_Q G(P,t;Q,s)| &\le C\frac{\delta(P)}{(t-s)^{n+2/2}} e^{-c|P-Q|^2/(t-s)}.\end{aligned}$$

*Case* 2. $\delta(Q) \ge (t-s)^{1/2}$. The proof is the same as in Case 1 except that we apply the Schauder interior estimate to $G(P,t;\cdot,\cdot)$ in $Q'_R = B(Q,R) \times (s, s+R^2)$ with $R = (t-s)^{1/2}/2$, and use (ii) of Lemma 1.3 to control $G(P,t;Q',S')$, $(Q',s') \in Q'_R$.

The proof of (ii) and (iii) is similar to the proof of (i). We omit the details.

## 2. The heat equation in a cylinder

**2.1. The initial Dirichlet problem.** In this subsection we will follow the argument in [DK2] to prove that all strong solutions $u(x,t)$ are in one-to-one correspondence with suitable pairs of measures $\mu$ on $D$ and $\lambda$ on $\partial D$. We will also show that any strong solution $u$ has an explicit representation given by

$$u(x,t) = \int_{Q\in D} G(x,t;Q,0)\,d\mu(Q) + \int_{Q\in\partial D} \frac{\partial G}{\partial N_Q}(x,t;Q,0)\,d\lambda(Q)$$

where $\partial/\partial N_Q$ is the derivative in the direction of the inward normal $N_Q$ at $Q \in \partial D$.

We will start with some definitions and results of [DK2]. We say that $u$ is a strong solution of the initial Dirichlet problem (IDP) for the equation $\Delta u = \partial_t u$ in $D\times(0, \infty)$ if $u$ is a continuous, nonnegative function in $\overline{D}\times(0, \infty)$, $u \equiv 0$ on $\partial D \times (0, \infty)$ and for all smooth functions $\eta$ on $\overline{D} \times (0, \infty)$ which vanish on $\partial D \times [\tau_1, \tau_2]$, $\tau_1 > 0$, we have

$$\iint_{D\times[\tau_1, \tau_2]} \left[u\Delta\eta + u\frac{\partial\eta}{\partial t}\right] dx\,dt$$
$$= \int_D u(x, \tau_2)\eta(x, \tau_2)\,dx - \int_D u(x, \tau_1)\eta(x, \tau_1)\,dx.$$

**Lemma 2.1.1** [DK2, Lemma 2]. *Let $u_1$, $u_2$ be strong solutions of the (IDP) in $D \times (0, \infty)$. Let $\eta \geq 0$, $\eta \in C_0^\infty(D)$, and let $T < \infty$, $\{\tau_j\} \searrow 0$, $\tau_0 = T$. Then there exist nonnegative measures $\{\lambda_j\}$ on $D$ with $\int_D d\lambda_j \leq \int \eta\,dx$ and*

$$\int_D [w_1(x, T) - w_2(x, T)]\eta(x)\,dx = \int_D w_1(x, \tau_j) - w_2(x, \tau_j)\,d\lambda_j(x)$$

*where $w_i(x) = Gu_i(x) = \int_D G(x, y)u_i(y)\,dy$ (i.e. the Green's potential of $u_i$), $i = 1, 2$.*

*Proof.* The proof is contained in the proof of Lemma 2.12 of [DK1].

Let $u$ be a strong solution of the (IDP) and let $w = Gu$. Then $\partial w/\partial t = -u \leq 0$ (see (2.25) of [DK1]) and hence $\lim_{t\to 0} w(x, t)$ exists for each $x \in D$.

**Lemma 2.1.2** (Pierre's maximum principle). *Let $u_1$, $u_2$ be two strong solutions of the (IDP) in $D\times(0, \infty)$. Suppose that $u_2 \in C(\overline{D}\times[0, \infty))$ and that $w_i = Gu_i$ verify $\lim_{t\to 0} w_1(x, t) \geq \lim_{t\to 0} w_2(x, t)$. Then $w_1(x, t) \geq w_2(x, t)$.*

*Proof.* Same as the proof of Lemma 3 of [DK2].

**Lemma 2.1.3.** *Let $u$ be a strong solution of the (IDP) in $D \times (0, \infty)$ and $w = Gu$. Then there exists $x_0 \in D$ such that $\lim_{t\to 0} w(x_0, t) < \infty$.*

*Proof.* Suppose that $\lim_{t\to 0} w(x, t) = \infty \ \ \forall x \in D$. We claim that this implies that $w(x, t) = \infty \ \ \forall x \in D, \ t > 0$. In order to prove the claim we first note that Lemma 2.1.2 implies that

$$w(x, t) \geq w_f(x, t) = Gu_f \quad \forall f \in C_0^\infty(D)$$

where $u_f$ is the solution of $\Delta u_f = \partial_t u_f$ in $D \times (0, \infty)$ with initial data $u_f(x, 0) = f(x)$, $\forall x \in D$ and boundary data $u_f|_{\partial D\times(0,\infty)} \equiv 0$.

$$\Rightarrow w(x, t) \geq w_{\lambda f}(x, t) = Gu_{\lambda f} = G(\lambda u_f)$$
$$= \lambda Gu_f \quad \forall\lambda > 0, \ f \in C_0^\infty.$$

Now fix an $f \in C_0^\infty$, $f \geq 0$, and $f(\overline{x}) > 0$ for some $\overline{x} \in D$. Then

$$u_f(x, t) > 0 \quad \forall x \in D, \ t > 0$$

by the Harnack inequality. So $Gu_f(x, t) > 0$ and thus

$$w(x, t) \geq \lambda Gu_f(x, t) \to \infty \quad \text{as } \lambda \to \infty.$$

On the other hand $\forall x \in D$, $t > 0$,

$$w(x, t) = \int_D G(x, y)u(y, t)\,dy \leq \|G(x, \cdot)\|_{L^1(D)}\|u(\cdot, t)\|_{L^\infty(D)} < \infty.$$

Contradiction arises. Therefore there must exist an $x_0 \in D$ such that

$$\lim_{t\to 0} w(x_0, t) < \infty.$$

**Lemma 2.1.4.** *Let $u$ be a strong solution of the (IDP) in $D \times (0, \infty)$. Then*

$$\sup_{t>0} \int_D u(x, t)\delta(x)\,dx < \infty \quad \text{and} \quad \int_D h(x)\,dx < \infty$$

*where $h(x) = \lim_{t\to 0} w(x, t)$.*

*Proof.* The proof follows the argument used in the proof of Lemma 6 in [DK2].

**Theorem 2.1.5.** *Let $u$ be a strong solution of the (IDP). Then there is a positive Borel measure $\mu$ on $D$ with $\int_D \delta(x)\,d\mu(x) < \infty$ and a positive Borel measure $\lambda$ on $\partial D$ with $\int_{\partial D} d\lambda < \infty$ such that whenever $\eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$, we have*

$$\lim_{t\to 0} \int_D u(x, t)\eta(x)\,dx = \int_D \eta\,d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N}\,d\lambda$$

*where $\partial/\partial N$ is the derivative in the direction of the inward normal $N$.*

*Proof.* Let $h(x) = \lim_{t\to 0} w(x, t)$. By Lemma 6.1.4, $\int_D h(x)\,dx < \infty$. Hence $h$ is superharmonic in $D$ and we can use the same argument as in the proof of Theorem 7 in [DK2] to finish the proof of Theorem 2.1.5.

**Theorem 2.1.6** (Uniqueness). *Suppose $u_1$ and $u_2$ are two strong solutions of the (IDP) and that*

$$\lim_{t\to 0} \int_D u_1(x, t)\eta(x)\,dx = \lim_{t\to 0} \int_D u_2(x, t)\eta(x)\,dx \qquad \forall \eta \in C^\infty(R^n),\ \eta|_{\partial D} \equiv 0.$$

*Then $u_1 \equiv u_2$.*

*Proof.* The same as the proof of Theorem 8 of [DK2].

We need one more lemma whose proof is not hard.

**Lemma 2.1.7.** *Let $G(x, t; Q, s)$ be the Green kernel for the heat equation and $\eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$. Then*

$$\int_{x\in D} G(x, t; Q, 0)\eta(x)\,dx \leq C\delta(Q) < \infty \quad \forall Q \in D,\ 0 < t \leq 1,$$

*for some constant $C$ independent of $Q \in D$ and $0 < t \leq 1$ where $\delta(Q) =$* $\mathrm{dist}(Q, \partial D)$.

**Theorem 2.1.8** (Existence). *Given a pair of measures $\mu$ on $D$ and $\lambda$ on $\partial D$ with $\mu \geq 0$, $\lambda \geq 0$, $\int_D \delta(x)\,d\mu(x) < \infty$, $\int_{\partial D} d\lambda < \infty$, there is a unique strong solution $u$ of the (IDP) such that for $\eta \in C^\infty(R^n)$, $\eta|_{\partial D} \equiv 0$, we have*

$$\lim_{t\to 0} \int_D u(x, t)\eta(x)\,dx = \int_D \eta\,d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N}\,d\lambda$$

*where $\partial/\partial N$ is the derivative in the direction of the inward normal $N$. In fact $u(x, t)$ is given explicitly by*

$$u(x, t) = \int_{Q\in D} G(x, t; Q, 0)\, d\mu(Q) + \int_{Q\in\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)\, d\lambda(Q)$$

*where $G(x, t; Q, s)$ is the Green kernel for the heat equation.*

*Proof*. Let

$$\begin{aligned} u_1(x, t) &= \int_{Q\in D} G(x, t; Q, 0)\, d\mu(Q), \\ u_2(x, t) &= \int_{Q\in\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)\, d\lambda(Q). \end{aligned}$$

By the uniqueness theorem, it suffices to show that both $u_1$ and $u_2$ are strong solutions of (IDP) and that $u_1 + u_2$ has initial trace $\mu$ on $D$ and $\lambda$ on $\partial D$. Clearly both $u_1$ and $u_2$ are continuous, nonnegative functions on $D\times(0, \infty)$. Since

$$\begin{aligned} \left|\int_{Q\in D} G(x, t; Q, 0)\, d\mu(Q)\right| &\le C\int_D \frac{\delta(Q)}{t^{n+1/2}} e^{-c|x-Q|^2/t}\, d\mu(Q) \\ &\le \frac{C}{t^{n+1/2}}\int_D \delta(Q)\, d\mu(Q) \end{aligned}$$

and

$$\begin{aligned} \int_{\partial D}\left|\frac{\partial G}{\partial N_q}(x, t; Q, 0)\right| d\lambda(Q) &\le \int_{\partial D} \frac{C}{t^{n+1/2}} e^{-c|x-Q|^2/t}\, d\lambda(Q) \\ &\le \frac{C}{t^{n+1/2}}\int_{\partial D} d\lambda(Q) < \infty, \end{aligned}$$

we have $\forall\psi\in C^\infty(\overline{D}\times(0, \infty))$, $\psi|_{\partial D\times[\tau_1, \tau_2]} \equiv 0$, $\tau_1 > 0$,

$$\begin{aligned} &\iint_{D\times[\tau_1, \tau_2]} \left[u_1\Delta\psi + u_1\frac{\partial\psi}{\partial t}\right] dx\, dt \\ &= \int_D \iint_{D\times[\tau_1, \tau_2]} G(x, t; Q, 0)\left[\Delta\psi + \frac{\partial\psi}{\partial t}\right] dx\, dt\, d\mu(Q) \\ &= \int_D \left(\iint_{D\times[\tau_1, \tau_2]} \left[\Delta_x G(x, t; Q, 0)\psi(x, t)\right.\right. \\ &\qquad \left.\left. -\frac{\partial G}{\partial t}(x, t; Q, 0)\psi(x, t)\right] dx\, dt\right) d\mu(Q) \\ &\quad + \int_{Q\in D}\int_D G(x, t; Q, 0)\psi(x, t)\, dx\Big|_{t=\tau_1}^{t=\tau_2} d\mu(Q) \\ &= \int_D u_1(x, \tau_2)\psi(x, \tau_2)\, dx - \int_D u_1(x, \tau_1)\psi(x, \tau_1)\, dx \end{aligned}$$

and

$$\begin{aligned}
&\iint_{D\times[\tau_1,\tau_2]}\left[u_2\Delta\psi+u_2\frac{\partial\psi}{\partial t}\right]dx\,dt\\
&=\int_{\partial D}\iint_{D\times[\tau_1,\tau_2]}\frac{\partial G}{\partial N_Q}(x,t;Q,0)\left[\Delta\psi+\frac{\partial\psi}{\partial t}\right]dx\,dt\,d\lambda(Q)\\
&=\int_{\partial D}\iint_{D\times[\tau_1,\tau_2]}\left[\frac{\partial}{\partial N_Q}\Delta_x G(x,t;Q,0)\psi(x,t)\right.\\
&\qquad\qquad\left.-\frac{\partial}{\partial N_Q}\partial_t G(x,t;Q,0)\psi(x,t)\right]dx\,dt\,d\lambda(Q)\\
&\quad+\int_{\partial D}\int_D\frac{\partial G}{\partial N_Q}(x,t;Q,0)\psi(x,t)\,dx\Big|_{t=\tau_1}^{t=\tau_2}d\lambda(Q)\\
&=\int_D u_2(x,\tau_2)\psi(x,\tau_2)\,dx-\int_D u_2(x,\tau_1)\psi(x,\tau_1)\,dx.
\end{aligned}$$

It remains to show that

(i) $\lim_{(x,t)\to(Q_0,t_1)} u_1(x,t)=\lim_{(x,t)\to(Q_0,t_1)} u_2(x,t)=0\quad\forall Q_0\in\partial D,\ t_1>0,$

and (ii): $u_1$ has trace $\mu$ on $D$, 0 on $\partial D$, and $u_2$ has trace 0 on $D$, $\lambda$ on $\partial D$.

*Proof of* (i).

$$\begin{aligned}
u_1(x,t)&=\int_{Q\in D}G(x,t;Q,0)\,d\mu(Q)\\
&=\int_{Q\in D}\frac{G(x,t;Q,0)}{\delta(Q)}\delta(Q)\,d\mu(Q).
\end{aligned}$$

By Lemma 1.3,

$$G(x,t;Q,0)\le C_T\frac{\delta(x)\delta(Q)}{t^{n+2/2}}e^{-c|x-Q|^2/t}\quad\text{for } 0<t<T$$

for any $T>0$. The Lebesgue dominated convergence theorem then implies that $u_1(x,t)\to 0$ as $(x,t)\to(Q_0,t_1)$, $Q_0\in\partial D$, $t_1>0$. That is,

$$\lim_{(x,t)\to(Q_0,t_1)}u_1(x,t)=0\ \forall Q_0\in\partial D,\ t_1>0.$$

Similarly $u_2(x,t)\to 0$ as $(x,t)\to(Q_0,t_1)$, $\forall Q_0\in\partial D$, $t_1>0$. Hence $u_1$ and $u_2$ are both strong solutions of the (IDP).

*Proof of* (ii). By Fubini's theorem for any $\eta\in C^\infty(R^n)$, $\eta|_{\partial D}\equiv 0$,

$$\begin{aligned}
\int_{x\in D}u_1(x,t)\eta(x)\,dx&=\int_{x\in D}\int_{Q\in D}G(x,t;Q,0)\,d\mu(Q)\eta(x)\,dx\\
&=\int_{Q\in D}\left(\int_{x\in D}G(x,t;Q,0)\eta(x)\,dx\right)d\mu(Q)\\
&\to\int_{Q\in D}\eta\,d\mu\quad\text{by Lemma 1.7}.
\end{aligned}$$

Also

$$\begin{aligned}\int_D u_2(x,t)\eta(x)\,dx &= \int_{Q\in\partial D}\int_{x\in D}\frac{\partial G}{\partial N_Q}(x,t;Q,0)\eta(x)\,dx\,d\lambda(Q)\\ &= \int_{Q\in\partial D}\frac{\partial}{\partial N_Q}\left(\int_{x\in D}G(x,t;Q,0)\eta(x)\,dx\right)d\lambda(Q)\\ &\to \int_{Q\in\partial D}\frac{\partial\eta}{\partial N_Q}(Q)\,d\lambda(Q)\quad \text{as } t\to 0.\end{aligned}$$

**2.2. A priori estimate.** For any point $S\in\partial D$, there exists a local $C^\infty$ space coordinate system $\phi_S: R^n\to R$ at $S$, i.e., there exists $r_0>0$ such that

$$\begin{aligned}D\cap B(S,r_0) &= \{Q=(Q',Q_n): Q'\in R^{n-1},\ Q_n\in R,\ Q_n>\phi_S(Q')\}\\ &\quad\cap B(S,r_0),\\ \partial D\cap B(S,r_0) &= \{Q=(Q',Q_n): Q'\in R^{n-1},\ Q_n\in R,\ Q_n=\phi_S(Q')\}\\ &\quad\cap B(S,r_0)\end{aligned}$$

and with the $\{Q_n=0\}$ plane being tangential to $\partial D$ at $S$ and the origin of the local space coordinate system is at $S$, i.e., $S=(0,0)$ in the local space coordinate at $S$.

Since $\partial D\in C^\infty$, $\partial D$ satisfies the interior cone condition at $S=(0,0)$, i.e., there exists $M>0$, $\alpha>0$, such that

$$\Gamma(S)=\{Q=(Q',Q_n): |Q'|\le MQ_n,\ 0<Q_n\le\alpha\}\subset D$$

where $(Q',Q_n)$ is the local space coordinate of $Q$ with respect to the coordinate system $(S,\phi_S)$.

We define the nontangential approach to $(S,0)\in\partial D\times\{0\}$ in $D\times(0,\infty)$ to be the space time cone

$$\begin{aligned}\Gamma_\beta(S)=\{(Q,t)=(Q',Q_n,t)\in R^{n-1}\times R^+\times R^+ :&\\ |Q'|\le MQ_n,\ Q_n=\beta t^{3/2}\le\alpha\}&\end{aligned}$$

where $(Q',Q_n)$ is the local space coordinate of $Q$ with respect to the coordinate system $(S,\phi_S)$.

Note that we can choose $M$ and $\alpha$ such that they are independent of $\phi_S$ and $S\in\partial D$ but depend only on $\partial D$ and $r_0$.

**Lemma 2.2.1.** *$\forall\alpha>0$, $a>0$, there exists $C_\alpha=C(a,\alpha)>0$ such that*

$$e^{-a|x-z|^2/t}\le C_\alpha e^{-a|x|^2/(2t)}\quad \forall x\in R^n,\ t>0,\ |z|\le\alpha\sqrt{t}.$$

**Lemma 2.2.2.** *Fix an $S\in\partial D$ and let $\phi=\phi_S: R^{n-1}\to R$ be the $C^\infty$ local coordinate system associated with $S$ as described at the beginning of this section, i.e. there exists $r_0>0$ such that*

$$\begin{aligned}D\cap B(S,r_0) &= \{Q=(Q',Q_n): Q'\in R^{n-1}, Q_n\in R, Q_n>\phi_S(Q')\}\\ &\quad\cap B(S,r_0),\\ \partial D\cap B(S,r_0) &= \{Q=(Q',Q_n): Q'\in R^{n-1}, Q_n\in R, Q_n=\phi_S(Q')\}\\ &\quad\cap B(S,r_0).\end{aligned}$$

*Let $\nu$ be a positive Borel measure on $D$ with $\int_D \delta(x)\,d\nu(x) < \infty$ and* $\operatorname{supp}\nu \subset B(S, r_0)$. *If $u(x, t) = \int_D G(x, t; Q, 0)\,d\nu(Q)$, then*

$$u(x, t) \le CM(\delta(Q)\,d\nu(Q))(Q_0) \quad \forall (x, t) \in \Gamma_\beta(Q_0), \ Q_0 \in B(S, r_0) \cap \partial D,$$

*where $C$ is independent of $Q_0 \in B(S, r_0) \cap \partial D$ and $\nu$.*

$$M(f)(Q_0) = \sup_{\rho>0} \frac{1}{|B'(Q_0', \rho)|} \iint_{\substack{|Q'-Q_0'|\le\rho \\ Q=(Q', Q_n),\ Q'\in R^{n-1}}} f(Q)\,dQ,$$

$$M(\delta(Q)\,d\nu(Q))(Q_0) = \sup_{\rho>0} \frac{1}{|B'(Q_0', \rho)|} \iint_{\substack{|Q'-Q_0'|\le\rho \\ Q=(Q', Q_n),\ Q'\in R^{n-1}}} \delta(Q)\,d\nu(Q)$$

*where $Q_0 = (Q_0', \phi(Q_0')) \in \partial D \cap B(S, r_0)$, $Q = (Q', Q_n)$ is the local space coordinate of $Q$ with respect to the local coordinate system $(S, \phi_S)$ at $S$ as described at the beginning of this section, and $B'(Q_0', \rho) = \{Q' \in R^{n-1} : |Q' - Q_0'| \le \rho\}$.*

*Proof*. By Lemma 1.3,

$$G(x, t; Q, 0) \le C\frac{\delta(x)\delta(Q)}{t^{n+2/2}} e^{-c|x-Q|^2/t}.$$

So for any $(x, t) \in \Gamma_\beta(Q_0)$, $Q_0 \in B(S, r_0) \cap \partial D$,

$$\begin{aligned}
u(x, t) &= \int_{Q\in D} G(x, t; Q, 0)\,d\nu(Q) \\
&\le C\frac{\delta(x)}{t^{3/2}} \cdot \frac{1}{t^{n-1/2}} \iint_{Q\in D} e^{-c|x-Q|^2/t} \delta(Q)\,d\nu(Q) \\
&\le C' \frac{1}{t^{n-1/2}} \iint_{Q\in D} e^{-c'|Q_0-Q|^2/t} \delta(Q)\,d\nu(Q) \quad \text{by Lemma 2.2.1} \\
&\le C' \frac{1}{t^{n-1/2}} \Bigg\{ \iint_{\substack{|Q'-Q_0'|\le t^{1/2} \\ Q\in D}} e^{-c'|Q_0'-Q'|^2/t} \delta(Q)\,d\nu(Q) \\
&\qquad + \sum_{k=0}^{\infty} \iint_{\substack{2^k t^{1/2}\le|Q'-Q_0'|\le 2^{k+1}t^{1/2} \\ Q\in D}} e^{-c'|Q_0'-Q'|^2/t} \delta(Q)\,d\nu(Q) \Bigg\} \\
&\le C\left(1 + \sum_{k=0}^{\infty} (2^{k+1})^{n-1} e^{-c' 2^{2k}}\right) M(\delta(Q)\,d\nu(Q))(Q_0) \\
&\le C' M(\delta(Q)\,d\nu(Q))(Q_0).
\end{aligned}$$

**Theorem 2.2.3.** *Let $u_2(x, t) = \int_{\partial D} \partial G/\partial N_Q(x, t; Q, 0)\,d\lambda(Q)$ where $\lambda$ is a positive Borel measure on $\partial D$ with $\int_{\partial D} d\lambda < \infty$. Then $u_2(x, t) \le C\widetilde{M}(d\lambda)(Q_0)$ for all $(x, t) \in \Gamma_\beta(Q_0)$, $Q_0 \in \partial D$, where*

$$\widetilde{M}(d\lambda)(Q_0) = \sup_{\rho>0} \frac{1}{\rho^{n-1}} \int_{\substack{|Q-Q_0|\le\rho \\ Q\in\partial D}} d\lambda(Q), \qquad Q_0 \in \partial D.$$

*Proof*. By Lemma 1.3,

$$\frac{\partial G}{\partial N_Q}(x, t; Q, 0) \le C\frac{\delta(x)}{t^{3/2}} \cdot \frac{1}{t^{n-1/2}} e^{-c|x-Q|^2/t}.$$

So

$$\begin{aligned} u_2(x, t) &\le C\frac{\delta(x)}{t^{3/2}} \cdot \frac{1}{t^{n-1/2}} \int_{Q\in\partial D} e^{-c|x-Q|^2/t}\, d\lambda(Q) \\ &\le C\frac{1}{t^{n-1/2}} \int_{Q\in\partial D} e^{-c|x-Q|^2/t}\, d\lambda(Q) \quad \text{for } (x, t) \in \Gamma_\beta(Q_0) \\ &\le C'\widetilde{M}(d\lambda)(Q_0) \quad \forall (x, t) \in \Gamma_\beta(Q_0),\ Q_0 \in \partial D. \end{aligned}$$

**2.3. A Fatou theorem for the corner points of a cylinder.** In this subsection we will prove the convergence a.e. on $\partial D \times \{0\}$ of the strong solution $u(x, t)$ of the heat equation in a cylinder as $(x, t) \to (Q_0, 0) \in \partial D \times \{0\}$ along the space time cone $\Gamma_\beta(Q_0)$ as defined in subsection 2.2 by assuming the everywhere convergence of the solution $\overline{u}$ of the heat equation at the corner points $\partial D \times \{0\}$ with initial trace $(d\mu, d\lambda) = (0, d\sigma)$ (where $d\sigma$ is the surface measure on $\partial D$) along these nontangential paths. We will use the same notation as in subsection 2.2 throughout this subsection.

**Lemma 2.3.1.** *With the same notation as Lemma* 2.2.2, *there exists* $C > 0$ *such that*

$$|\{Q_0 \cap \partial D \cap B(S, r_0) : \lambda \le M(\delta(Q)\, d\nu(Q))(Q_0)\}| \le \frac{C}{\lambda} \int_{B(S, r_0)\cap D} \delta(Q)\, d\nu(Q).$$

*Proof*. See [S, Chapter 1].

**Theorem 2.3.2.** *Let* $u_1 = \int_D G(x, t; Q, 0)\, d\mu(Q)$ *with* $\mu$ *being a positive Borel measure on* $D$ *with* $\int_D \delta(Q)\, d\mu(Q) < \infty$. *Then*

$$\lim_{\substack{(x, t)\in\Gamma_\beta(Q_0) \\ t\to 0}} u_1(x, t) = 0 \quad \textit{a.e. } Q_0 \in \partial D.$$

*Proof*. Note that in order to prove the theorem, it suffices to show that

$$\lim_{\substack{(x, t)\in\Gamma_\beta(Q_0) \\ t\to 0}} u_1(x, t) = 0 \quad \text{a.e. } Q_0 \in \partial D \cap B(S, r_0)$$

for any $S \in \partial D$ and $r_0$ is as described at the beginning of this subsection.

Hence without loss of generality, we fix an $S \in \partial D$ and assume from now on that $Q_0 \in \partial D \cap B(S, r_0)$. Write

$$E_r = B(S, r_0) \cap \{Q \in D : \delta(Q) \le r\}.$$

Then for all $(x, t) \in \Gamma_\beta(Q_0)$,

$$\begin{aligned} u_1(x, t) &= \int_{E_r^c\cap D} G(x, t; Q, 0)\, d\mu(Q) + \int_{E_r} G(x, t; Q, 0)\, d\mu(Q) \\ &\le C\int_{Q\in E_r^c\cap D} \frac{\delta(x)\delta(Q)}{t^{n+2/2}} e^{-c|x-Q|^2/t}\, d\mu(Q) \\ &\quad + CM(\delta(Q)\chi_{E_r}\, d\mu(Q))(Q_0) \\ &= I_1 + I_2 \end{aligned}$$

by Lemma 1.3 and Lemma 2.2.2 where $\chi_A$ is the characteristic function of the set $A$.

By the Lebesgue dominated convergence theorem, $I_1$ vanishes as $(x, t)$ $(\in \Gamma_\beta(Q_0)) \to (Q_0, 0)$. Hence for any $\lambda > 0$,

$$\begin{aligned}
&\left|\left\{Q_0 \in \partial D \cap B(S, r_0) : \lambda \le \varlimsup_{\substack{(x,t)\in\Gamma_\beta(Q_0)\\ t\to 0}} u_1(x, t)\right\}\right| \\
&\le |\{Q_0 \in \partial D \cap B(S, r_0) : \lambda \le CM(\delta(Q)\chi_{E_r}\, d\mu(Q))(Q_0)\}| \\
&\le \frac{C}{\lambda}\int_{Q\in B(S, r_0)\cap D} \delta(Q)\chi_{E_r}\, d\mu(Q) \\
&\le \frac{C}{\lambda}\int_{\substack{Q\in B(S, r_0)\cap D\\ \delta(Q)\le r}} \delta(Q)\, d\mu(Q) \to 0 \quad \text{as } r \to 0
\end{aligned}$$

since $\int_{B(S, r_0)\cap D} \delta(Q)\, d\mu(Q) \le \int_D \delta(Q)\, d\mu(Q) < \infty$. Hence

$$\left|\left\{Q_0 \in \partial D \cap B(S, r_0) : \lambda \le \varlimsup_{\substack{(x,t)\in\Gamma_\beta(Q_0)\\ t\to 0}} u_1(x, t)\right\}\right| = 0 \quad \forall\lambda > 0.$$

Therefore

$$\lim_{\substack{(x,t)\in\Gamma_\beta(Q_0)\\ t\to 0}} u_1(x, t) = 0 \quad \text{a.e. } Q_0 \in \partial D.$$

**Lemma 2.3.3.** *Let $\lambda$ be a positive Borel measure on $\partial D$ with $\int_{\partial D} d\lambda < \infty$ and let*

$$u_2(x, t) = \int_{\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)\, d\lambda(Q).$$

*Then for almost every $Q_0 \in \partial D$ we have*

$$\lim_{\substack{(x,t)\in\Gamma_\beta(Q_0)\\ t\to 0}} \left|u_2(x, t) - f(Q_0)\int_{\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)\, d\sigma(Q)\right| = 0$$

*where $d\sigma$ is the surface measure on $\partial D$ and $f = d\lambda/d\sigma$ is the Radon-Nikodym derivative of $d\lambda$ with respect to $d\sigma$. (Note that $\int_{\partial D} f\, d\sigma \le \int_{\partial D} d\lambda < \infty$ and $f \ge 0$ a.e. $[d\sigma]$.)*

*Proof*. Write $d\lambda = f\, d\sigma + d\nu_s$ where $d\nu_s \perp d\sigma$. We will assume from now on that $(x, t) \in \Gamma_\beta(Q_0)$ in the proof. Now since $d\nu_s \perp d\sigma$,

$$\lim_{r\to 0} \frac{1}{r^{n-1}} \int_{\substack{|Q-Q_0|\le r\\ Q\in\partial D}} |d\nu_s|(Q) = 0 \quad \text{a.e. } Q_0 \in \partial D.$$

Also a.e. $Q_0 \in \partial D$ are Lebesgue points of $f$. Let $Q_0 \in \partial D$ be a Lebesgue point of $f$ satisfying

$$\lim_{r\to 0} \frac{1}{r^{n-1}} \int_{\substack{|Q-Q_0|\le r\\ Q\in\partial D}} |d\nu_s|(Q) = 0 \quad \text{and} \quad 0 \le f(Q_0) < \infty.$$

Then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\frac{1}{r^{n-1}} \int_{\substack{|Q-Q_0|\le r \\ Q\in\partial D}} |d\nu_s|(Q) \le \varepsilon$$

and

$$\frac{1}{r^{n-1}} \int_{\substack{|Q-Q_0|\le r \\ Q\in\partial D}} |f(Q) - f(Q_0)|\, d\sigma(Q) \le \varepsilon \quad \forall 0 < r \le \delta .$$

Choose $k_0 \in Z^+$, $k_0 = k_0(t)$, such that $2^{k_0} t^{1/2} \le \delta < 2^{k_0+1} t^{1/2}$. Then

$$\left| u_2(x, t) - f(Q_0) \int_{\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)\, d\sigma(Q) \right|$$

$$\le \int_{\substack{|Q-Q_0|\le t^{1/2} \\ Q\in\partial D}} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)(|f(Q) - f(Q_0)|\, d\sigma(Q) + |d\nu_s|(Q))$$

$$+ \sum_{k=0}^{k_0} \int_{\substack{2^k t^{1/2} \le |Q-Q_0| \le 2^{k+1} t^{1/2} \\ Q\in\partial D}} \frac{\partial G}{\partial N_Q}(x, t; Q, 0) \cdot (|f(Q) - f(Q_0)|\, d\sigma(Q) + |d\nu_s|(Q))$$

$$+ \int_{\substack{|Q-Q_0|\ge \delta \\ Q\in\partial D}} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)((f(Q) - f(Q_0))\, d\sigma(Q) + d\nu_s(Q)) .$$

Since $|x - Q| = |x - Q_0 + Q_0 - Q|$ and $(x, t) \in \Gamma_\beta(Q_0) \Rightarrow \delta(x) \le |x - Q_0| \le Ct^{3/2} \le C\sqrt{t}$, by Lemma 2.2.1 and Lemma 1.3 the last term above is dominated by

$$C \frac{\delta(x)}{t^{n+2/2}} \int_{\substack{|Q-Q_0|\ge \delta \\ Q\in\partial D}} e^{-c|x-Q|^2/t} (d\lambda(Q) + |f(Q_0)|\, d\sigma(Q))$$

$$\le \frac{C'}{t^{n-1/2}} \int_{\substack{|Q-Q_0|\ge \delta \\ Q\in\partial D}} e^{-c|Q_0-Q|^2/t} (d\lambda(Q) + |f(Q_0)|\, d\sigma(Q))$$

$$\le \frac{C}{t^{n-1/2}} e^{-c'\delta^2/t} \left| \int_{\partial D} d\lambda(Q) + |f(Q_0)|\, d\sigma(Q) \right| \to 0 \quad \text{as } t \to 0 .$$

While the first two terms are dominated by

$$
\begin{aligned}
& C\frac{\delta(x)}{t^{3/2}} \cdot \frac{1}{t^{n-1/2}} \int_{\substack{|Q-Q_0|\le t^{1/2} \\ Q\in\partial D}} e^{-c|x-Q|^2/t}(|f(Q)-f(Q_0)|\,d\sigma(Q)+|d\nu_s|(Q)) \\
& \quad + C\frac{\delta(x)}{t^{3/2}} \cdot \sum_{k=0}^{k_0} \frac{(2^{k+1})^{n-1}}{(2^{k+1}t^{1/2})^{n-1}} \cdot \int_{\substack{2^k t^{1/2}\le|Q-Q_0|\le 2^{k+1}t^{1/2} \\ Q\in\partial D}} e^{-c|x-Q|^2/t} \\
& \qquad\qquad \cdot (|f(Q)-f(Q_0)|\,d\sigma(Q)+|d\nu_s|(Q)) \\
& \le \frac{C}{t^{n-1/2}} \int_{\substack{|Q-Q_0|\le t^{1/2} \\ Q\in\partial D}} (|f(Q)-f(Q_0)|\,d\sigma(Q)+|d\nu_s|(Q)) \\
& \quad + C\sum_{k=0}^{k_0} \frac{(2^{k+1})^{n-1}}{(2^{k+1}t^{1/2})^{n-1}} \cdot \int_{\substack{|Q-Q_0|\le 2^{k+1}t^{1/2} \\ Q\in\partial D}} e^{-c|x-Q|^2/t} \\
& \qquad\qquad \cdot (|f(Q)-f(Q_0)|\,d\sigma(Q)+|d\nu_s|(Q)) \\
& \le C\left(1+\sum_{k=0}^{\infty} 2^{(k+1)(n-1)}e^{-c'2^{2k}}\right)\cdot\varepsilon \le C'\varepsilon .
\end{aligned}
$$

Therefore

$$\varlimsup_{\substack{(x,t)\in\Gamma_\beta(Q_0) \\ t\to 0}} \left|u_2(x,t)-f(Q_0)\int_{\partial D}\frac{\partial G}{\partial N_Q}(x,t;Q,0)\,d\sigma(Q)\right| \le C'\varepsilon$$

for all $\varepsilon>0$.

$$\Rightarrow \lim_{\substack{(x,t)\in\Gamma_\beta(Q_0) \\ t\to 0}} \left|u_2(x,t)-f(Q_0)\int_{\partial D}\frac{\partial G}{\partial N_Q}(x,t;Q,0)\,d\sigma(Q)\right| = 0$$

and the proof is completed.

**Theorem 2.3.4.** *Let $\overline{u}(x,t)=\int_{\partial D}\partial G/\partial N_Q(x,t;Q,0)\,d\sigma(Q)$ with $d\sigma$ as the surface measure on $\partial D$ and $\Gamma_\beta(Q_0)$ as in subsection 2.2. Then*

$$\lim_{\substack{(x,t)\in\Gamma_\beta(Q_0) \\ t\to 0}} \overline{u}(x,t)=\frac{\beta}{\sqrt{4\pi}} \quad \forall Q_0\in\partial D .$$

*Proof.* Same as the proof of Theorem 2.4.1 of subsection 2.4.

**Theorem 2.3.5.** *Let $u(x,t)$ be a strong solution of the (IDP) in $D\times(0,\infty)$ with initial trace $\mu$ on $D$ and $\lambda$ on $\partial D$. (See Theorem 2.1.8 for its existence.) Then*

$$\lim_{\substack{(x,t)\in\Gamma_\beta(Q_0) \\ t\to 0}} u(x,t)=\frac{\beta}{\sqrt{4\pi}}\cdot\frac{d\lambda}{d\sigma}(Q_0) \quad \text{a.e. } Q_0\in\partial D$$

*where $d\lambda/d\sigma$ is the Radon-Nikodym derivative of $d\lambda$ with respect to the surface measure $d\sigma$ on $\partial D$ and $\Gamma_\beta(Q_0)$ is as defined in subsection 2.2.*

*Proof*. By Theorems 2.1.5, 2.1.6, 2.1.8, $u(x, t) = u_1(x, t) + u_2(x, t)$ where

$$u_1(x, t) = \int_{Q\in D} G(x, t; Q, 0)\, d\mu(Q),$$
$$u_2(x, t) = \int_{Q\in\partial D} \frac{\partial G}{\partial N_Q}(x, t; Q, 0)\, d\lambda(Q).$$

By Theorem 2.3.2,

$$\lim_{\substack{(x, t)\in\Gamma_\beta(Q_0)\\ t\to 0}} u_1(x, t) = 0 \quad \text{a.e. } Q_0 \in \partial D.$$

By Theorem 2.3.3 and Theorem 2.3.4,

$$\lim_{\substack{(x, t)\in\Gamma_\beta(Q_0)\\ t\to 0}} u_2(x, t) = \frac{\beta}{\sqrt{4\pi}} \cdot \frac{d\lambda}{d\sigma}(Q_0) \quad \text{a.e. } Q_0 \in \partial D.$$

Therefore

$$\begin{aligned}\lim_{\substack{(x, t)\in\Gamma_\beta(Q_0)\\ t\to 0}} u(x, t) &= \lim_{\substack{(x, t)\in\Gamma_\beta(Q_0)\\ t\to 0}} u_1(x, t) + \lim_{\substack{(x, t)\in\Gamma_\beta(Q_0)\\ t\to 0}} u_2(x, t)\\ &= \frac{\beta}{\sqrt{4\pi}} \cdot \frac{d\lambda}{d\sigma}(Q_0)\end{aligned}$$

for a.e. $Q_0 \in \partial D$.

**2.4. A Fatou theorem for the solution with initial trace** $(0, d\sigma)$**.** We will start this section with a definition. We let

$$\begin{aligned}\overline{\Gamma}_\beta(Q_0) = \{(Q, s) = (Q', Q_n, s) \in R^{n-1} \times R^+ \times R^- : |Q'| \le MQ_n,\\ Q_n = \beta(-s)^{3/2} \le \alpha\}\end{aligned}$$

where $Q_0 \in \partial D$ and $Q = (Q', Q_n)$ is the local space coordinate of $Q \in D$ with respect to the coordinate system $\phi = \phi_{Q_0}$ at $Q_0$ as defined in subsection 2.2, $\alpha$ is the constant defined in the beginning of subsection 2.2, and $M$ is the Lipschitz constant for $\partial D$. The main result of this section is the following theorem.

**Theorem 2.4.1.** *Let* $v(Q, s) = \int_{\partial D} \partial G/\partial N_x(x, 0; Q, s)\, d\sigma(x)$, $s < 0$, *with* $d\sigma$ *being the surface measure on* $\partial D$. *Then*

$$\lim_{\substack{(Q, s)\in\overline{\Gamma}_\beta(Q_0)\\ s\to 0}} v(Q, s) = \frac{\beta}{\sqrt{4\pi}} \quad \forall Q_0 \in \partial D.$$

Before proving the theorem we would like to assume without loss of generality that $Q_0 = 0$ and $\phi$ is the local coordinate system at $Q_0 = 0$ with $\nabla\phi(0) = 0$, $\phi \in C_0^\infty(R^{n-1})$.

We first note that if $D'$ is the image of $D$ under the transformation $(x', x_n) \to (x', x_n')$ with $x_n' = x_n - \phi(x')$, $x = (x', x_n) \in D$ being the local coordinate of $x$ with respect to $(0, \phi)$, and if $\tilde{u}((x', x_n'), t) = u((x', x_n' + \phi(x')), t)$, $(x', x_n') \in D'$, then

$$\begin{aligned}&\Delta u - \partial_t u = 0 \quad \text{in } D \times (-\infty, \infty)\\ \Leftrightarrow\ &\operatorname{div}(a_{ij}(x')\nabla\tilde{u}) - \partial_t\tilde{u} = 0 \quad \text{in } D' \times (-\infty, \infty)\end{aligned}$$

where $a_{ij}(x') = \delta_{ij}$, $a_{1j}(x') = a_{j1} = -\partial\phi/\partial x_j$, $i, j = 1, \ldots, n-1$, $a_{nn} = 1 + |\nabla_{x'}\phi(x')|^2$. It is not hard to see that $A(x') = (a_{ij})$ is a uniformly elliptic matrix with the eigenvalues bounded above and below by positive constants independent of $x'$, $\forall(x', x_n) \in D'$, and depends on $\partial D$ only.

If $G(x, t; y, \tau) = G((x', x_n), t; (y', y_n), \tau)$ is the Green's function for the heat equation $\Delta - \partial_t = 0$ in $D \times (-\infty, \infty)$, then

$$\widetilde{G}((x', x_n'), t; (y', y_n'), \tau) = G((x', x_n' + \phi(x')), t; (y', y_n' + \phi(y')), \tau)$$

is the Green's function for $\operatorname{div}(a_{ij}\nabla) - \partial_t = 0$ in $D' \times (-\infty, \infty)$.

*Idea of proof of Theorem* 2.4.1. We first note that

$$\int_{x\in\partial D} \frac{\partial G}{\partial N_x}(x, 0; y, \tau)\, d\sigma(x) = \Bigg(\int_{\substack{x\in\partial D\\ |x'|>\varepsilon}} + \int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}}\Bigg) \frac{\partial G}{\partial N_x}(x, 0; y, \tau)\, d\sigma(x).$$

The first term vanishes as $(y, \tau)\ \ (\in \overline{\Gamma}_\beta(0)) \to 0$ by the Lebesgue dominated convergence theorem since its integrand

$$\begin{aligned}
&\le C\frac{\delta(y)}{|\tau|^{n+2/2}} \exp\left(-C\frac{|x-y|^2}{|\tau|}\right) \quad \text{by Lemma 1.3}\\
&\le \frac{C}{|\tau|^{n-1/2}} \exp\left(-C'\frac{\varepsilon^2}{|\tau|}\right) \quad \text{for } (y, \tau) \in \overline{\Gamma}_\beta(0),\ |y| \le \frac{\varepsilon}{2},\\
&\to 0 \quad \text{as } \tau \to 0.
\end{aligned}$$

While the second term

$$\begin{aligned}
&= \int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}} \frac{\partial G}{\partial N_x}(x, 0; y, \tau)\, d\sigma(x)\\
&= \int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}} \nabla_x G(x, 0; y, \tau)\cdot(-\nabla_{x'}\phi(x'), 1)\, dx'\\
&= -\int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}} \nabla_{x'} G(x, 0; y, \tau)\cdot\nabla_{x'}\phi(x')\, dx' + \int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}} \frac{\partial G}{\partial x_n}(x, 0; y, \tau)\, dx'.
\end{aligned}$$

The term

$$\begin{aligned}
&\left|\int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}} \nabla_{x'} G\cdot\nabla_{x'}\phi\, dx'\right|\\
&\qquad\le C \sup_{|x'|\le\varepsilon}\{|\nabla^2_{x'}\phi(x')|\,|x'|\}\cdot \int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}} \frac{\delta(y)}{|\tau|^{n+2/2}} \exp\left(-C\frac{|x-y|^2}{|\tau|}\right) dx'\\
&\qquad\le C \sup_{|x'|\le\varepsilon} |\nabla^2_{x'}\phi(x')|\cdot\varepsilon\cdot \int_{|x'|\le\varepsilon} \frac{1}{|\tau|^{n-1/2}} \exp\left(-C'\frac{|x'|^2}{|\tau|}\right) dx'\\
&\qquad\le C'\varepsilon.
\end{aligned}$$

Therefore it remains to consider

$$\int_{\substack{x\in\partial D\\|x'|\le\varepsilon}}\frac{\partial G}{\partial x_n}(x,0;y,\tau)\,dx'=\int_{\substack{x\in\partial D\\|x'|\le\varepsilon}}\frac{\partial\widetilde{G}}{\partial x_n'}(x,0;y,\tau)\,dx'$$

and show that it converges to $\beta/\sqrt{4\pi}$ as $(y,\tau)$ $(\in\overline{\Gamma}_\beta(0))\to 0$. To show this we note that $\widetilde{G}(x,t;y,s)=G_y(x,t;y,s)+E(x,t;y,s)$ where

$$E(x,t;y,s)=\int_s^t d\lambda\int_{D'}G_\eta(x,t;\eta,\lambda)\Phi(\eta,\lambda;y,s)\,d\eta.$$

(See Lemma 2.4.6 below.) $G_y(x,t;z,s)$ is the Green's function for

$$L^y u=\sum a_{ij}(y')\frac{\partial^2 u}{\partial x_i\partial x_j}-\frac{\partial u}{\partial t}=0,\quad y=(y',y_n'),$$

from which $\partial\widetilde{G}/\partial x_n=\partial G_y/\partial x_n+\partial E/\partial x_n$. Since $x\in\partial D'$, $|x'|\le\varepsilon\le r_0\Rightarrow x_n'=0$, $d\sigma|_{\partial D'\cap\{|x|\le\varepsilon\}}=dx'$. If

$$\int_{\substack{x\in\partial D\\|x'|\le\varepsilon}}|\nabla_x E|\,d\sigma(x)\to 0\quad\text{as } (y,\tau)\ (\in\overline{\Gamma}_\beta(0))\to 0,$$

then

$$\lim_{\substack{(y,\tau)\in\overline{\Gamma}_\beta(0)\\ \tau\to 0}}\int_{\substack{x\in\partial D'\\|x'|\le\varepsilon}}\frac{\partial\widetilde{G}}{\partial x_n}\,dx'=\lim_{\substack{(y,\tau)\in\overline{\Gamma}_\beta(0)\\ \tau\to 0}}\int_{\substack{x\in\partial D'\\|x'|\le\varepsilon}}\frac{\partial G_y}{\partial x_n}\,dx'.$$

On the other hand, layer potential method implies that the function

$$\Rightarrow w_y(x,0;y,\tau)=\sum a_{ij}(y)n_i(x)\frac{\partial G_y}{\partial x_j}(x,0;y,\tau)$$

satisfies

$$\begin{aligned}w_y(x,0;y,\tau)&=2\sum a_{ij}(y)n_i(x)\frac{\partial Z_y}{\partial x_j}(x,0;y,\tau)\\&\quad-2\int_\tau^0 d\lambda\int_{\eta\in\partial D'}H_y(x,0;\eta,\lambda)w_y(\eta,\lambda;y,\tau)\,d\sigma(\eta)\end{aligned}$$

where $Z_y(x,t;\eta,\tau)$ is the fundamental solution for $L^y u=0$ and

$$\begin{aligned}H_y(x,t;\eta,\lambda)&=\sum a_{ij}(y)n_i(x)\frac{\partial Z_y}{\partial x_j}(x,t;\eta,\lambda),\\(n_i(x))_{i=1}^n&=N_x=\text{unit inward normal at } x\in\partial D'\\&=(0,\dots,0,1)\quad\text{if } |x|\le\varepsilon,\ x\in\partial D'.\end{aligned}$$

So

$$\begin{aligned}\sum a_{ij}(y)n_i(x)\frac{\partial G_y}{\partial x_j}(x,0;y,\tau)&=\sum_j a_{nj}(y)\frac{\partial G_y}{\partial x_j}(x,0;y,\tau)\\&=\sum_{1\le j\le n-1}-\frac{\partial\phi}{\partial y_j}(y')\cdot\frac{\partial G_y}{\partial x_j}+\frac{\partial G_y}{\partial x_n}.\end{aligned}$$

Therefore

$$\left|\int_{\substack{x\in\partial D'\\|x'|\le\varepsilon}}\sum_{1\le j\le n-1}-\frac{\partial\phi}{\partial y_j}(y')\frac{\partial G_y}{\partial x_j}(x,0;y,\tau)\,dx'\right|$$
$$\le C\sup_{|x'|\le\varepsilon}|\nabla\phi(x')|\cdot\int_{\substack{x\in\partial D'\\|x'|\le\varepsilon}}\frac{\delta(y)}{|\tau|^{n+2/2}}\exp\left(-C\frac{|x-y|^2}{|\tau|}\right)dx'\le C'\varepsilon$$

as before. Hence

$$\begin{aligned}\int_{x\in\partial D}\frac{\partial G}{\partial N_x}\,d\sigma(x)&\approx\int_{\substack{x\in\partial D'\\|x'|\le\varepsilon}}\frac{\partial G}{\partial x_n}\,dx=\int_{\substack{x\in\partial D'\\|x'|\le\varepsilon}}\frac{\partial\widetilde{G}}{\partial x'_n}\,dx'\\
&\approx\int_{|x'|\le\varepsilon}\frac{\partial G_y}{\partial x'_n}\,dx'+\int_{|x'|\le\varepsilon}\frac{\partial E}{\partial x'_n}\,dx'\\
&\approx\int_{|x'|\le\varepsilon}\frac{\partial G_y}{\partial x'_n}\,dx'\approx\int_{|x'|\le\varepsilon}\sum a_{ij}(y)n_i(x)\frac{\partial G_y}{\partial x_j}(x,0;y,\tau)\,dx'\\
&=2\int_{|x'|\le\varepsilon}\sum a_{ij}(y)n_i(x)\frac{\partial Z_y}{\partial x_j}(x,0;y,\tau)\,dx'\\
&\quad-2\int_{|x'|\le\varepsilon}\int_\tau^0 d\lambda\int_{\eta\in\partial D'}H_y(x,0;\eta,\lambda)w_y(\eta,\lambda;y,\tau)\,d\sigma(\eta)\,dx'\\
&\to\frac{\beta}{\sqrt{4\pi}}\end{aligned}$$

if the first term above $\to\beta/\sqrt{4\pi}$ and the second term above $\to 0$ as $(y,\tau)$ $(\in\overline{\Gamma}_\beta(0))\to 0$.

By the above discussion, we see that in order to prove Theorem 2.4.1, it is natural to first investigate properties of $\widetilde{G}$. We would let $G_y(x,t;\eta,\tau)$ be the Green's function for the equation

$$L^y u=\sum a_{ij}(y)\frac{\partial^2 u}{\partial x_i\partial x_j}(x,t)-\frac{\partial u}{\partial t}(x,t)=0\quad\text{in }\partial D'\times(-\infty,\infty)$$

and

$$Z_y(x,t;\eta,\lambda)=\frac{1}{(\det a^{ij}(y))^{1/2}}\cdot\frac{1}{(4\pi(t-\tau))^{n/2}}\exp\left\{-\frac{\sum a^{ij}(y)(x_i-\eta_i)(x_j-\eta_j)}{4(t-\tau)}\right\}$$

be its fundamental solution throughout this section where $(a_{ij}(x))$ is as defined on page 624 and $(a^{ij})=(a_{ij})^{-1}$. We will also assume without loss of generality that $\partial D'$ is smooth throughout this section. Since the adjoint equation for $L^y$ is

$$L^{y^*}v=\sum a_{ij}(y)\frac{\partial^2 v}{\partial z_i\partial z_j}(z,\tau)+\frac{\partial v}{\partial t}(z,\tau)=0,$$

maximum principle holds for solutions of $L^y$ and $L^{y^*}$. Therefore Lemma 1.3 and Lemma 1.4 remains valid for $G_y$. For the sake of completeness, we state them here again.

**Lemma 2.4.2.** *For* $-T \le \tau < t \le T$, $T > 0$, *we have*

(i)

$$G_y(x, t; z, \tau) \le \frac{C}{(t-\tau)^{n/2}} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right), \qquad x, z \in D',$$

(ii)

$$G_y(x, t; z, \tau) \le C\frac{\delta(z)}{(t-\tau)^{n+1/2}} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right), \qquad x, z \in D',$$

(iii)

$$G_y(x, t; z, \tau) \le C\frac{\delta(x)\delta(z)}{(t-\tau)^{n+2/2}} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right), \qquad x, z \in D',$$

(iv)

$$\left|\frac{\partial G_y}{\partial N_x}(x, t; z, \tau)\right| \le C\frac{\delta(z)}{(t-\tau)^{n+2/2}} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right), \qquad x \in \partial D', \; z \in D',$$

*where* $\partial/\partial N_x$ *is the derivative in the direction of the normal to* $\partial D'$ *at the point* $x \in \partial D'$, $\delta(x) = \operatorname{dist}(x, \partial D')$, *and* $C$ *is a constant independent of* $x, z, t, \tau$.

**Corollary 2.4.3.** *The following is true:*

(i) $$|\nabla_z G_y(x, t; z, \tau)| \le C\frac{\delta(x)}{(t-\tau)^{n+2/2}} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right),$$

(ii) $$|\nabla_x G_y(x, t; z, \tau)| \le C\frac{\delta(z)}{(t-\tau)^{n+2/2}} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right)$$

*for all* $-T \le \tau < t \le T$ (*for any fixed* $T > 0$), $x, z \in \overline{D'}$, *and* $C$ *is a constant independent of* $y$ *and depending only on* $T > 0$.

The following lemma comes from [LSU, Theorem 16.3].

**Lemma 2.4.4.** *The Green's function* $G_y(x, t; z, \tau)$ *for*

$$L^y = \sum a_{ij}(y)\partial^2_{x_ix_j} - \partial_t = 0 \quad \text{in } D' \times (-\infty, \infty)$$

*satisfies the following inequalities:*

(i) $$|\partial_t^r \partial_x^s G_y(x, t; z, \tau)| \le \frac{C}{(t-\tau)^{n+2r+s/2}} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right),$$

(ii) $$\begin{aligned}&|\partial_t^r \partial_x^s G_y(x, t; z, \tau) - \partial_{t'}^r \partial_x^s G_y(x, t'; z, \tau)| \\ &\quad \le C(t-t')^{3/2-r-s/2}(t'-\tau)^{-(n+3)/2} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right),\end{aligned}$$

*where* $2r + s = 1, 2$ *and* $\tau < t' < t$, *and*

(iii) $$\begin{aligned}&|\partial_t^r \partial_x^s G_y(x, t; z, \tau) - \partial_t^r \partial_{\overline{x}}^s G_y(\overline{x}, t; z, \tau)| \\ &\quad \le C\frac{|x-\overline{x}|}{(t-\tau)^{n+3/2}} \exp\left(-C\frac{|x''-z|^2}{t-\tau}\right)\end{aligned}$$

*where* $2r + s = 2$ *and* $\overline{x}$ *is that one of the points* $x, x'$ *which is closest to* $y$, $C$ *is a positive constant independent of* $y$.

The following lemma comes from [F, Chapter 1, Lemma 3].

**Lemma 2.4.5.** *If* $-\infty < \alpha < n/2+1$, $-\infty < \beta < n/2+1$, *then*

$$\int_\sigma^t \int_{R^n} (t-\lambda)^{-\alpha} \exp\left(-\frac{h|x-\eta|^2}{4(t-\lambda)}\right)(\lambda-\sigma)^{-\beta}\exp\left(-\frac{h|\eta-y|^2}{4(\lambda-\sigma)}\right) d\eta\, d\lambda$$
$$= \left(\frac{4\pi}{h}\right)^{n/2} B\left(\frac{n}{2}-\alpha+1, \frac{n}{2}-\beta+1\right)(t-\sigma)^{n/2+1-\alpha-\beta}\exp\left(-\frac{h|x-y|^2}{4(t-\sigma)}\right)$$

*where* $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, $\Gamma$ *is the gamma function.*

**Lemma 2.4.6.** *If* $\widetilde{G}(x, t; y, \tau)$ *is the Green's function for* $Lu = \operatorname{div}(a_{ij}(x')\nabla u) - \partial_t u = 0$ *in* $D' \times (-\infty, \infty)$, $(x', x_n') \in D'$. *Then the following integral identity holds:*

$$\widetilde{G}(x, t; y, \tau) = G_y(x, t; y, \tau) + E(x, t; y, \tau),$$
$$E(x, t; y, \tau) = \int_\tau^t d\lambda \int_{D'} G_\eta(x, t; \eta, \lambda)\Phi(\eta, \lambda; y, \tau)\, d\eta$$

*with* $\Phi$ *being determined from the integral identity*

$$\Phi(x, t; y, \tau) = LG_y(x, t; y, \tau)$$
$$+ \int_\tau^t d\lambda \int_{D'} LG_\eta(x, t; \eta, \lambda)\Phi(\eta, \lambda; y, \tau)\, d\eta.$$

*Proof*. The proof is exactly the same as the construction of the fundamental solution of $Lu = 0$ given in [F, Chapter 1] except that we use Lemma 2.4.4 to control $G_y$ and replace Theorem 1 of Chapter 1 of [F] by the following lemma. We omit the details of its proof.

**Lemma 2.4.7.** *Let* $f(y, \tau)$ *be a continuous bounded function in* $D' \times (T_0, T_1)$. *Then* $J(x, t, \tau) = \int_{D'} G_y(x, t; y, \tau) f(y, \tau)\, dy$ *is a continuous function in* $(x, t, \tau)$, $x \in \overline{D'}$, $T_0 \le \tau < t \le T_1$ *and* $\lim_{\tau\to t} J(x, t, \tau) = f(x, t)$.

**Corollary 2.4.8.** *Let* $v_R$ *be the solution of the adjoint equation*

$$L^{y^*} v_R = \sum a_{ij}(y')\frac{\partial^2 v_R}{\partial z_i \partial z_j}(z, \tau) + \frac{\partial v_R}{\partial \tau}(z, \tau) = 0 \quad \text{in } D_R \times (-2R^2, 0)$$

*of* $L^y$ *where* $D_R = B(0, 2R)\backslash\overline{B(0, R)}$ *with boundary value* $\phi(z/R, \tau/R^2)$, $\phi \in C^\infty$, $0 \le \phi \le 1$, *and*

$$\begin{cases} \phi(z, \tau) \equiv 1 & \text{for } (z, \tau) \in \partial B(0, 2) \times (-4, 0] \cup (B(0, 2)\backslash\overline{B(0, 1)}) \times \{0\}, \\ \phi(z, \tau) \equiv 0 & \text{for } (z, \tau) \in \partial B(0, 1) \times (-2, -4). \end{cases}$$

*Then*

$$\|\nabla_z v_R\|_{L^\infty(D_R \times (-4R^2, 0))} \le C/R < \infty$$

*for some constant* $C$ *independent of* $y \in D'$ *and* $R > 0$.

*Proof*. The lemma follows from the boundary Schauder's estimate. (See [F, Theorem 6, p. 65].)

**Lemma 2.4.9.** *With the same notation as in Lemma* 2.4.6, *then the function*

$$F(x, t; z, \tau) = LG_y(x, t; z, \tau)$$
$$= \operatorname{div}(a_{ij}(x')\nabla_x G_y(x, t; z, \tau)) - \partial_t G_y(x, t; z, \tau)$$

*satisfies the adjoint equation* $L^{y^*}F = 0$ *where*

$$L^{y^*} = \sum a_{ij}(y')\frac{\partial^2}{\partial z_i \partial z_j} + \frac{\partial}{\partial \tau}.$$

$$F(x, t; z, \tau)|_{\substack{z\in\partial D'\\ t>\tau}} \equiv 0$$

*and*

(i) $$|F(x, t; z, \tau)| \le C\left\{\frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}}\right\}\cdot \exp\left\{-C\frac{|x-z|^2}{t-\tau}\right\},$$

(ii) $$|F(x, t; z, \tau)| \le C_\alpha\left\{\frac{|x-y|\delta(z)^{1-\alpha}}{(t-\tau)^{(n+3-\alpha)/2}} + \frac{\delta(z)^{1-\alpha}}{(t-\tau)^{(n+2-\alpha)/2}}\right\}\cdot \exp\left\{-C\frac{|x-z|^2}{t-\tau}\right\}$$

*for some constants* $C_\alpha, C > 0$, $C_\alpha, C$ *both independent of* $y \in \partial D'$, $-1 < \tau < t < 1$, $x, z \in \partial D'$, $0 \le \alpha < 1$.

*Proof*. We first note that

$$L^{y^*}F(x, t; z, \tau) = L^{y^*}LG_y(x, t; z, \tau) = 0$$

and

$$\begin{aligned} &G_y(x, t; z, \tau)|_{\substack{z\in\partial D'\\ t>\tau}} = 0\\ &\Rightarrow \partial_x^r\partial_t^s G_y(x, t; z, \tau)|_{\substack{z\in\partial D'\\ t>\tau}} = 0 \quad \forall r, s = 0, 1, 2, \dots\\ &\Rightarrow F(x, t; z, \tau) = LG_y(x, t; z, \tau)|_{\substack{z\in\partial D'\\ t>\tau}} = 0.\end{aligned}$$

Also

$$\begin{aligned} &|F(x, t; z, \tau)|\\ &\quad = |LG_y(x, t; z, \tau)|\\ &\quad = \left|\sum a_{ij}(x')\frac{\partial^2 G_y}{\partial x_i \partial x_j}(x, t; z, \tau) - \Delta_{y'}\phi(y')\frac{\partial G_y}{\partial x_n}(x, t; z, \tau) - \frac{\partial G_y}{\partial t}(x, t; z, \tau)\right|\\ &\quad = \left|\sum (a_{ij}(x') - a_{ij}(y'))\frac{\partial^2 G_y}{\partial x_i \partial x_j}(x, t; z, \tau) - \Delta_{y'}\phi(y')\frac{\partial G_y}{\partial x_n}(x, t; z, \tau)\right|\\ &\quad \le C\frac{|x-y|}{(t-\tau)^{n+2/2}}\exp\left(-C\frac{|x-z|^2}{t-\tau}\right)\\ &\qquad + C\frac{1}{(t-\tau)^{n+1/2}}\cdot\exp\left(-C\frac{|x-z|^2}{t-\tau}\right).\end{aligned}$$

*Proof of* (ii). The proof of (ii) of Lemma 2.4.9 will be similar to the proof of (ii) of Lemma 1.4. We first note that since $\partial D'$ is smooth, $\partial D'$ satisfies the exterior sphere condition, i.e. there exists a positive constant $h > 0$ such that for each $x \in \partial D'$, $\forall 0 < r \le h$, there exists $x_0 \in D'^c$ such that $\overline{B(x_0, r)} \cap \overline{D'} = \{x\}$.

*Case* 1. $\delta(z) \ge h$, then

$$1 \le \frac{\delta(z)^{1-\alpha}}{h^{1-\alpha}} \le \frac{\delta(z)^{1-\alpha}}{h^{1-\alpha}}\cdot\frac{(t-\tau)^{1-\alpha/2}}{(t-\tau)^{1-\alpha/2}} \le C_\alpha\frac{\delta(z)^{1-\alpha}}{(t-\tau)^{1-\alpha/2}}$$

since $|t|, |\tau| \le 1$. So by (i),

$$
\begin{aligned}
&|F(x, t; z, \tau)| \\
&\quad \le C_\alpha \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right) \frac{\delta(z)^{1-\alpha}}{(t-\tau)^{1-\alpha/2}} \\
&\quad \le C'_\alpha \left\{ \frac{|x-y|\delta(z)^{1-\alpha}}{(t-\tau)^{n+3-\alpha/2}} + \frac{\delta(z)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}} \right\} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right).
\end{aligned}
$$

*Case* 2. $\delta(z) \ge (t-\tau)^{1/2}/4$. The proof is similar to the proof of Case 1.

*Case* 3. $\delta(z) \le h$ and $\delta(z) \le (t-\tau)^{1/2}/4 \le h$.

We fix $z^* \in \partial D'$ with $|z - z^*| = \delta(z)$ and set $R = (t-\tau)^{1/2}/4$. Since $\partial D'$ satisfies the exterior sphere condition, there exists $z_0 \in \partial D'^c$ such that $\overline{B(z_0, R)} \cap \overline{D'} = \{z^*\}$. We may also assume $\overline{zz^*} \subset D'$. Consider the cylinder $Q_R = D_R \times (\tau, \tau + (2R)^2)$, $D_R = B(z_0, 2R) \backslash \overline{B(z_0, R)}$. Then, we have $(z, \tau) \in \overline{Q}_R \cap \overline{D'} \times (-1, 1)$. Similar to the proof of Case 3 of Lemma 1.4, for any $(\overline{z}, \overline{\tau}) \in \overline{Q}_R \cap \overline{D'} \times [-1, 1]$

$$
|F(x, t; \overline{z}, \overline{\tau})| \le C \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp\left(-C'\frac{|x-z|^2}{t-\tau}\right).
$$

If $\overline{v}_R(\overline{z}, \overline{\tau}) = v_R(\overline{z} - z_0, \overline{\tau} - (\tau + 4R^2))$, $(\overline{z}, \overline{\tau}) \in \overline{Q}_R$, where $v_R$ is as in Corollary 2.4.8, then since

$$
\begin{cases} F(x, t; \overline{z}, \overline{\tau}) = 0, \\ \overline{v}_R(\overline{z}, \overline{\tau}) \ge 0 \quad \forall (\overline{z}, \overline{\tau}) \in \overline{Q}_R \cap \partial D' \times (-\infty, \infty) \end{cases}
$$

and

$$
\begin{cases} |F(x, t; \overline{z}, \overline{\tau})| \le C \left\{ \dfrac{|x-y|}{(t-\tau)^{n+2/2}} + \dfrac{1}{(t-\tau)^{n+1/2}} \right\} \exp\left(-C\dfrac{|x-z|^2}{t-\tau}\right), \\ \overline{v}_R(\overline{z}, \overline{\tau}) \equiv 1 \end{cases}
$$

for all

$$
\begin{aligned}
(\overline{z}, \overline{\tau}) \in \{&\partial B(z_0, 2R) \times (\tau, \tau + (2R)^2] \\
&\cup (B(z_0, 2R) \backslash B(z_0, R)) \times \{\tau + (2R)^2\}\} \cap \overline{D} \times (-\infty, \infty),
\end{aligned}
$$

by the maximum principle applied to the functions $F(x, t, \cdot, \cdot)$ and

$$
C \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right) \overline{v}_R(\cdot, \cdot)
$$

in the region $Q_R \cap D \times (-\infty, \infty)$, we have

$$
\begin{aligned}
|F(x, t; z, \tau)| &\le C \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right) \overline{v}_R(z, \tau) \\
&\le C \frac{\delta(z)}{(t-\tau)^{1/2}} \left\{ \frac{|x-y|}{(t-\tau)^{n+2/2}} + \frac{1}{(t-\tau)^{n+1/2}} \right\} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right) \\
&= C_\alpha \left\{ \frac{|x-y|\delta(z)^{1-\alpha}}{(t-\tau)^{n+3-\alpha/2}} + \frac{\delta(z)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}} \right\} \exp\left(-C\frac{|x-z|^2}{t-\tau}\right)
\end{aligned}
$$

(since $\delta(z) \le (t-\tau)^{1/2}/4 \Rightarrow (\delta(z)/(t-\tau)^{1/2})^\alpha \le 4^{-\alpha}$).

*Case* 4. $\delta(z) \le h$ and $\delta(z) \le h \le 4h \le (t-\tau)^{1/2}$.

Proof of Case 4 is similar to Case 3. We omit the details.

**Lemma 2.4.10.** *With the same notation as* 2.4.6, *the function* $\Phi(x, t; y, \tau)$ *satisfies the inequality*

$$|\Phi(x, t; y, \tau)| \le C_\alpha \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}} \exp\left(-C\frac{|x-y|^2}{t-\tau}\right), \qquad 0 < \alpha < 1,$$

*where* $C_\alpha$ *is a constant independent of* $x, y \in D'$ *and* $t, \tau$, $-1 < \tau < t < 1$.

*Proof*. Since $\Phi$ satisfies the integral identity,

$$\Phi(x, t; y, \tau) = LG_y(x, t; y, \tau) + \int_\tau^t d\lambda \int_{\partial D'} LG_\eta(x, t; \eta, \lambda)\Phi(\eta, \lambda; y, \tau)\, d\eta.$$

Therefore $\Phi(x, t; y, \tau) = \sum_{\nu=1}^{\infty}(LG_y)_\nu(x, t; y, \tau)$ where $(LG_y)_1(x, t; y, \tau) = LG_y(x, t; y, \tau)$ and

$$(LG_y)_{\nu+1}(x, t; y, \tau) = \int_\tau^t d\lambda \int_{\partial D'} LG_\eta(x, t; \eta, \lambda)(LG_y)_\nu(\eta, \lambda; y, \tau)\, d\eta$$

for all $\nu = 1, 2, \dots$. By Lemma 2.4.9,

$$\begin{aligned} |LG_y(x, t; y, \tau)| \ (= |F(x, t; y, \tau)|) &\le C_\alpha \left\{\frac{|x-y|\delta(y)^{1-\alpha}}{(t-\tau)^{n+3-\alpha/2}} + \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}}\right\} \exp\left(-C\frac{|x-y|^2}{t-\tau}\right) \\ &\le C'_\alpha \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}} \exp\left(-C'\frac{|x-y|^2}{t-\tau}\right). \end{aligned}$$

Also from the proof of Lemma 2.4.9,

$$\begin{aligned} |LG_\eta(x, t; \eta, \lambda)| &\le C\frac{|x-\eta|}{(t-\lambda)^{n+2/2}} \exp\left(-C\frac{|x-\eta|^2}{t-\lambda}\right) \\ &\quad + C\frac{1}{(t-\lambda)^{n+1/2}} \exp\left(-C\frac{|x-\eta|^2}{t-\lambda}\right) \\ &\le C'\frac{1}{(t-\lambda)^{n+1/2}} \exp\left(-C\frac{|x-\eta|^2}{t-\lambda}\right). \end{aligned}$$

Therefore

$$\begin{aligned} |(LG_y)_2(x, t; y, \tau)| &\le C_\alpha \int_\tau^t d\lambda \int_{D'} \frac{1}{(t-\lambda)^{n+1/2}} \exp\left(-C\frac{|x-\eta|^2}{t-\lambda}\right) \\ &\qquad \cdot \frac{\delta(y)^{1-\alpha}}{(\lambda-\tau)^{n+2-\alpha/2}} \exp\left(-C\frac{|\eta-y|^2}{\lambda-\tau}\right) d\eta \\ &= C_\alpha \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+1-\alpha/2}} \exp\left(-C\frac{|x-y|^2}{t-\tau}\right) \frac{\Gamma(1/2)\Gamma(\alpha/2)}{\Gamma(\alpha+1/2)} \end{aligned}$$

by Lemma 2.4.5. And in general

$$\begin{aligned} &|(LG_y)_{\nu+1}(x, t; y, \tau)| \\ &\quad \le C_\alpha \frac{\Gamma(1/2)^\nu \Gamma(\alpha/2)}{\Gamma(\alpha+\nu/2)} \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+2-\nu-\alpha/2}} \exp\left(-C\frac{|x-y|^2}{t-\tau}\right) \end{aligned}$$

for all $\nu = 0, 1, 2, \ldots$. Therefore $\sum_{\nu=0}^{\infty}(LG_y)_{\nu+1}(x, t; y, \tau)$ is absolutely convergent and

$$\begin{aligned}|\Phi(x, t; y, \tau)| &\le \sum_{\nu=0}^{\infty} |(LG_y)_{\nu+1}(x, t; y, \tau)| \\ &\le C_\alpha \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+2-\alpha/2}} \exp\left(-C\frac{|x-y|^2}{t-\tau}\right).\end{aligned}$$

**Lemma 2.4.11.** *With the same notation as Lemma* 2.4.6, *then the function*

$$\begin{aligned}E(x, t; y, \tau) &= \widetilde{G}(x, t; y, \tau) - G_y(x, t; y, \tau) \\ &= \int_\tau^t d\lambda \int_{D'} G_y(x, t; \eta, \lambda)\Phi(\eta, \lambda : y, \tau)\, d\eta\end{aligned}$$

*satisfies the following inequality*

$$|\nabla_x E(x, t; y, \tau)| \le C_\alpha \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+1-\alpha/2}} \exp\left(-C\frac{|x-y|^2}{t-\tau}\right), \qquad 0 < \alpha < 1,$$

*where* $C_\alpha$ *is a constant independent of* $x, y \in D'$ *and* $t, \tau$, $-1 < \tau < t < 1$.

*Proof.* By Lemma 2.4.4,

$$|\nabla_x G_\eta(x, t; y, \tau)| \le \frac{C}{(t-\tau)^{n+1/2}} \exp\left(-C\frac{|x-y|^2}{t-\tau}\right).$$

So

$$\begin{aligned}|\nabla_x E(x, t; y, \tau)| &\le \int_\tau^t d\lambda \int_{D'} |\nabla_x G_y(x, t; \eta, \lambda)|\, |\Phi(\eta, \lambda : y, \tau)|\, d\eta \\ &\le C'_\alpha \int_\tau^t d\lambda \int_{D'} \frac{1}{(t-\lambda)^{n+1/2}} \exp\left(-C\frac{|x-\eta|^2}{t-\lambda}\right) \\ &\qquad \cdot \frac{\delta(y)^{1-\alpha}}{(\lambda-\tau)^{n+2-\alpha/2}} \exp\left(-C\frac{|\eta-y|^2}{\lambda-\tau}\right) d\eta \\ &\le C'_\alpha \frac{\delta(y)^{1-\alpha}}{(t-\tau)^{n+1-\alpha/2}} \exp\left(-C\frac{|x-y|^2}{t-\tau}\right)\end{aligned}$$

by Lemma 2.4.5 and Lemma 2.4.10.

**Lemma 2.4.12.** *With the same notation as Lemma* 2.4.6,

$$\lim_{\substack{(y, \tau)\in\overline{\Gamma}_\beta(0) \\ \tau\to 0}} \int_{\partial D'} |\nabla_x E(x, 0; y, \tau)|\, d\sigma(x) = 0.$$

*Proof.* Choose $0 < \alpha < 1/2$ in Lemma 2.4.11. Then for $(y, \tau) \in \overline{\Gamma}_\beta(0)$,

$$\begin{aligned}&\int_{\partial D'} |\nabla_x E(x, 0; y, \tau)|\, d\sigma(x) \\ &\quad\le C_\alpha \frac{\delta(y)^{1-\alpha}}{(-\tau)^{n+1-\alpha/2}} \int_{\partial D'} \exp\left(-C\frac{|x-y|^2}{-\tau}\right) d\sigma(x) \\ &\quad\le C'_\alpha \frac{(-\tau)^{3(1-\alpha)/2}}{(-\tau)^{n+1-\alpha/2}} \int_{\partial D'} \exp\left(-C'\frac{|x|^2}{-\tau}\right) d\sigma(x) \\ &\quad\le C_\alpha(-\tau)^{(1-2\alpha)/2} \to 0 \quad \text{as } \tau \to 0.\end{aligned}$$

**Lemma 2.4.13.** *The function*

$$\sum_{i,j} a_{ij}(y')n_i(x)\frac{\partial G_y}{\partial x_j}(x, t; y, \tau), \qquad x \in \partial D', \ y = (y', y'_n) \in D',$$

*where* $-1 < \tau < t < 1$ *and* $(n_i(x))_{i=1}^n = N_x$ *is the inward normal to* $\partial D'$ *at* $x$, *satisfies the following integral equation*:

$$\sum_{i,j} a_{ij}(y')n_i(x)\frac{\partial G_y}{\partial x_j}(x, t; y, \tau)$$
$$= 2H_y(x, t; y, \tau) - 2\int_\tau^t d\lambda \int_{\eta\in\partial D'} H_y(x, t; \eta, \lambda)w_y(\eta, \lambda; y, \tau)\, d\sigma(\eta)$$

*where*

$$w_y(\eta, \lambda; y, \tau) = \sum a_{ij}(y')n_i(\eta)\frac{\partial G_y}{\partial \eta_j}(\eta, \lambda; y, \tau),$$
$$H_y(x, t; \eta, \lambda) = \sum a_{ij}(y')n_i(x)\frac{\partial Z_y}{\partial x_j}(x, t; \eta, \lambda),$$
$$Z_y(x, t; \eta, \lambda) = \frac{1}{(\det(a^{ij}(y)))^{1/2}} \cdot \frac{1}{(4\pi(t-\lambda))^{n/2}} \exp\left\{-\frac{\sum a^{ij}(y)(x_i - \eta_i)(x_j - \eta_j)}{4(t-\tau)}\right\},$$
$$(a^{ij}(y)) = A^{-1} = (a_{ij}(y))^{-1}.$$

*Also*

$$\int_\tau^t \int_{\partial D'} |w_y(\eta, \lambda; y, \tau)|\, d\sigma(\eta) \le C < \infty, \quad C \textit{ independent of } y \in D'.$$

*Proof*. We first observe that $Z_y(x, t; \eta, \lambda)$ is the fundamental solution of

$$L^y u = \sum a_{ij}(y)\frac{\partial^2 u}{\partial x_i \partial x_j}(x, t) - \frac{\partial u}{\partial t}u(x, t) = 0.$$

By the theory of layer potentials [LSU, p. 409] and [PO] the Green's function $G_y(x, t; z, \tau)$ for the equation $L^y = 0$ in $D' \times (-\infty, \infty)$ is then given by

$$G_y(x, t; z, \tau) = Z_y(x, t; z, \tau) - g_y(x, t; z, \tau)$$

where

$$g_y(x, t; z, \tau) = \int_\tau^t d\lambda \int_{\partial D'} Z_y(x, t; \eta, \lambda)w_y(\eta, \lambda; z, \tau)\, d\sigma(\eta)$$

for all $x \in R^n$, $z \in D$, $\tau < t$, with the density $w_y$ being determined from the integral equation

$$w_y(x, t; z, \tau) = 2H_y(x, t; z, \tau) - 2\int_\tau^t d\lambda \int_{\partial D'} H_y(x, t; \eta, \lambda)w_y(\eta, \lambda; z, \tau)\, d\sigma(\eta).$$

Now for $x \in \partial D'$,

$$\lim_{\substack{\overline{x}(\in \overline{D'^c}) \to x \\ \overline{x}-x \perp N_x}} \sum a_{ij}(y) n_i(\overline{x}) \frac{\partial g_y}{\partial x_j}(\overline{x}, t; z, \tau) = H_y(x, t; z, \tau)$$

$$= W_y(x, t; z, \tau) + \frac{1}{2} w_y(x, t; z, \tau)$$

where

$$W_y(x, t; z, \tau) = \int_\tau^t d\lambda \int_{\partial D'} H_y(x, t; \eta, \lambda) w_y(\eta, \lambda; z, \tau)\, d\sigma(\eta)$$

and

$$\lim_{\substack{\overline{x}(\in \overline{D'}) \to x \\ \overline{x}-x \perp N_x}} \sum a_{ij}(y) n_i(\overline{x}) \frac{\partial g_y}{\partial x_j}(\overline{x}, t; z, \tau) = W_y(x, t; z, \tau) - \frac{1}{2} w_y(x, t; z, \tau)$$

$$= H_y(x, t; z, \tau) - w_y(x, t; z, \tau).$$

Therefore $\forall x \in \partial D'$,

$$\sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, t; z, \tau)$$

$$= \lim_{\substack{\overline{x}(\in D') \to x \\ \overline{x}-x \perp N_x}} \sum a_{ij}(y) n_i(\overline{x}) \frac{\partial G_y}{\partial x_j}(\overline{x}, t; z, \tau)$$

$$= \lim_{\substack{\overline{x}(\in D') \to x \\ \overline{x}-x \perp N_x}} \left\{ \sum a_{ij}(y) n_i(\overline{x}) \left( \frac{\partial Z_y}{\partial x_j}(\overline{x}, t; z, \tau) - \frac{\partial g_y}{\partial x_j}(\overline{x}, t; z, \tau) \right) \right\}$$

$$= \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, t; z, \tau) - (H_y(x, t; z, \tau) - w_y(x, t; z, \tau))$$

$$= w_y(x, t; z, \tau).$$

Also by the result of [LSU, p. 411], $\int_\tau^t d\lambda \int_{\partial D'} |w_y(\eta, \lambda; y, \tau)|\, d\sigma(\eta) \le C < \infty$ for some constant $C$ independent of $y$.

**Lemma 2.4.14.** *Use the same notation as Lemma* 2.4.6. *For any* $0 < 2\varepsilon \le r_0$ ($r_0$ *as in the beginning of subsection* 2.2),

$$2 \int_{\substack{|x| \le \varepsilon \\ x \in \partial D'}} H_y(x, 0; y, \tau)\, d\sigma(x) \to \frac{\beta}{\sqrt{4\pi}} \quad \text{as } (y, \tau) \in \overline{\Gamma_\beta}(0) \to (0, 0).$$

*Proof*. For $x \in \partial D'$, $|x| \le \varepsilon$, $(y, \tau) \in \overline{\Gamma_\beta}(0)$, $|y| \le \varepsilon$,

$$\sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, 0; y, \tau)$$

$$= \sum_{i,j,k} a_{ij}(y) n_i(x) \left( -\frac{a^{jk}(y)(x_k - y_k)}{2(-\tau)} \right) Z_y(x, 0; y, \tau)$$

$$= \frac{-\langle x - y, N_x \rangle}{2(-\tau)} Z_y(x, 0; y, \tau)$$

$$= \frac{y'_n}{2(-\tau)} Z_y(x, 0; y, \tau).$$

Therefore

$$
\begin{aligned}
2\int_{\substack{|x|\le\varepsilon\\ x\in\partial D'}} & H_y(x\,,0;y\,,\tau)\,d\sigma(x)\\
&= \frac{y_n-\phi(y')}{\sqrt{4\pi}(-\tau)^{3/2}}\cdot\frac{1}{(\det a^{ij}(y))^{1/2}(4\pi(-\tau))^{n-1/2}}\\
&\quad\cdot\int_{|x'|\le\varepsilon}\exp\left(-\frac{\sum a^{ij}(y)(x_i-y_i)(x_j-y_j)}{4(-\tau)}\right)dx'\\
&= \frac{y_n-\phi(y')}{\sqrt{4\pi}(-\tau)^{3/2}}\left(\frac{1}{(\det a^{ij}(y))^{1/2}}-1\right)\\
&\qquad\cdot\frac{1}{(4\pi(-\tau))^{n-1/2}}\int_{|x'|\le\varepsilon}\exp\left(-\frac{\sum a^{ij}(y)(x_i-y_i)(x_j-y_j)}{4(-\tau)}\right)dx'\\
&\qquad+\frac{y_n-\phi(y')}{\sqrt{4\pi}(-\tau)^{3/2}}\cdot\frac{1}{(4\pi(-\tau))^{n-1/2}}\\
&\qquad\cdot\int_{|x'|\le\varepsilon}\exp\left(-\frac{\sum a^{ij}(y)(x_i-y_i)(x_j-y_j)}{4(-\tau)}\right)dx'\\
&= I_1+I_2\,.
\end{aligned}
$$

$$
\begin{aligned}
|I_1| &\le \frac{|y_n|+|\phi(y')|}{\sqrt{4\pi}(-\tau)^{3/2}}\left|\frac{1}{(\det a^{ij}(y))^{1/2}}-1\right|\\
&\qquad\cdot\frac{1}{(4\pi(-\tau))^{n-1/2}}\int_{\substack{|x'|\le\varepsilon\\ x'\in R^{n-1}}}\exp\left(-C\frac{|x-y|^2}{(-\tau)}\right)dx'\\
&\le C\left|\frac{1}{(\det(a^{ij}(y')))^{1/2}}-1\right|\\
&\qquad\cdot\frac{1}{(4\pi(-\tau))^{n-1/2}}\int_{\substack{|x'|\le\varepsilon\\ x'\in R^{n-1}}}\exp\left(-C'\frac{|x'|^2}{(-\tau)}\right)dx'\\
&\le C'\left|\frac{1}{(\det(a^{ij}(y')))^{1/2}}-1\right|\to 0\quad\text{as } (y\,,\tau)\in\overline{\Gamma}_\beta(0)\to(0\,,0)
\end{aligned}
$$

since $(a^{ij}(y))^{-1}\to I=$ identity matrix as $y\to 0$.

For $I_2$, since $|y'|\le C(-\tau)^{3/2}$, we have

$$
\frac{y_n-\phi(y')}{(-\tau)^{3/2}}=\frac{y_n}{(-\tau)^{3/2}}+\frac{O(y'^2)}{(-\tau)^{3/2}}=\beta+O(\tau^{3/2})\to\beta\quad\text{as }\tau\to 0
$$

and

$$
\begin{aligned}
\frac{1}{(4\pi(-\tau))^{n-1/2}}&\int_{\substack{|x'|\le\varepsilon\\ x'\in R^{n-1}}}\exp\left(-\frac{\sum a^{ij}(y)(x_i-y_i)(x_j-y_j)}{4(-\tau)}\right)dx'\\
&\to\frac{1}{(4\pi)^{n-1/2}}\int_{\overline{x}'\in R^{n-1}}\exp(-|\overline{x}'|^2/4)\,d\overline{x}'=1\quad\text{as }\tau\to 0\,.
\end{aligned}
$$

Therefore $I_2 \to \beta/\sqrt{4\pi}$ as $(y, \tau) \in \overline{\Gamma}_\beta(0) \to (0, 0)$. So

$$2\int_{\substack{|x|\le\varepsilon \\ x\in\partial D'}} H_y(x, 0; y, \tau)\, d\sigma(x) = I_1 + I_2 \to \frac{\beta}{\sqrt{4\pi}} \quad \text{as } (y, \tau) \in \overline{\Gamma}_\beta(0) \to (0, 0).$$

**Lemma 2.4.15.** *With the same notation as Lemma* 2.4.13, *we have*

$$\begin{aligned}
&\int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\, dx' \\
&\quad = \int_{\substack{|x|\le\varepsilon \\ x'\in\partial D'}} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\, d\sigma(x) \\
&\quad \to \frac{\beta}{\sqrt{4\pi}} \quad \textit{as } (y, \tau) \in \overline{\Gamma}_\beta(0) \to (0; 0).
\end{aligned}$$

*Proof*. By Lemma 2.4.13,

$$\begin{aligned}
\sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau) &= 2H_y(x, 0; y, \tau) \\
&\quad - 2\int_\tau^0 d\lambda \int_{\eta\in\partial D'} H_y(x, 0; \eta, \lambda) w_y(\eta, \lambda; y, \tau)\, d\sigma(\eta).
\end{aligned}$$

Also $x_n' = 0$ for $x \in \partial D'$, $|x| \le \varepsilon$, therefore

$$\begin{aligned}
&\int_{\substack{|x|\le\varepsilon \\ x'\in\partial D'}} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\, d\sigma(x) \\
&\quad = \int_{\substack{|x|\le\varepsilon \\ x'\in\partial D'}} \sum a_{ij}(y) n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\, dx' \\
&\quad = \int_{\substack{|x|\le\varepsilon \\ x'\in\partial D'}} (-2) \int_\tau^0 d\lambda \left\{ \int_{\substack{|\eta|\le 2\varepsilon \\ \eta\in\partial D'}} + \int_{\substack{|\eta|> 2\varepsilon \\ \eta\in\partial D'}} \right\} \\
&\qquad\qquad \cdot H_y(x, 0; \eta, \lambda) w_y(\eta, \lambda; y, \tau)\, d\sigma(\eta)\, dx' \\
&\qquad + 2\int_{\substack{|x|\le\varepsilon \\ x'\in\partial D'}} H_y(x, 0; y, \tau)\, dx' \\
&\quad = J_1 + J_2 + J_3.
\end{aligned}$$

Now for $x, \eta \in \partial D'$, $|x|, |\eta| \le 2\varepsilon$, we have $x_n' = \eta_n' = 0$ and $N_x = (0, \ldots, 0, 1)$. So

$$\begin{aligned}
H_y(x, 0; \eta, \lambda) &= \sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, 0; \eta, \lambda) \\
&= \frac{-\langle N_x, x - \eta\rangle}{2(-\lambda)} Z_y(x, 0; \eta, \lambda) \\
&= 0.
\end{aligned}$$

Therefore $J_1 = 0$. For $|x| \le \varepsilon$, $|\eta| \ge 2\varepsilon$, $x, \eta \in \partial D'$, $(y, \tau) \in \overline{\Gamma}_\beta(0)$, $-1 < \tau < \lambda < 0$, since

$$\begin{aligned}
|H_y(x, 0; \eta, \lambda)| &= \left|\sum a_{ij}(y) n_i(x) \frac{\partial Z_y}{\partial x_j}(x, 0; \eta, \lambda)\right| \\
&= \left|\frac{-\langle N_x, x-\eta\rangle}{2(-\lambda)}\right| |Z_y(x, 0; \eta, \lambda)| \\
&\le C \frac{1}{(-\lambda)^{n+1/2}} \exp\left(-C\frac{\varepsilon^2}{(-\lambda)}\right).
\end{aligned}$$

Therefore $|J_2|$ is

$$\begin{aligned}
&\le C \int_{|x'|\le\varepsilon} \int_\tau^0 d\lambda \int_{\substack{\eta\in\partial D' \\ |\eta|\ge 2\varepsilon}} \frac{1}{(-\lambda)^{n+1/2}} \exp\left(-C\frac{\varepsilon^2}{(-\lambda)}\right) |w(\eta, \lambda, y, \tau)|\, d\sigma(\eta)\, dx' \\
&\le C\varepsilon^{n-1} \sup_{\tau\le\lambda\le 0} \frac{C}{|\lambda|^{n-1/2}} \exp\left(-C\frac{\varepsilon^2}{|\lambda|}\right) \to 0 \quad \text{as } (y, \tau)\ (\in \overline{\Gamma}_\beta(0)) \to (0, 0).
\end{aligned}$$

Also by Lemma 2.4.14, $J_3 \to \beta/\sqrt{4\pi}$ as $(y, \tau)\ (\in \overline{\Gamma}_\beta(0)) \to (0, 0)$. Therefore

$$\lim_{\substack{(y,\tau)\in\overline{\Gamma}_\beta(0) \\ \tau\to 0}} \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \sum a_{ij}(y') n_i(x) \frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\, dx' = \frac{\beta}{\sqrt{4\pi}}.$$

**Lemma 2.4.16.** *With the same notation as Lemma* 2.4.6 *and Lemma* 2.4.13, *we have*

$$\lim_{\substack{(y,\tau)\in\overline{\Gamma}_\beta(0) \\ \tau\to 0}} \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \frac{\partial \widetilde{G}}{\partial x_n'}(x, 0; y, \tau)\, dx' = \frac{\beta}{\sqrt{4\pi}}.$$

*Proof*. By Lemma 2.4.6,

$$\begin{aligned}
&\int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \frac{\partial \widetilde{G}}{\partial x_n'}(x, 0; y, \tau)\, dx' \\
&\quad = \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \frac{\partial G_y}{\partial x_n'}(x, 0; y, \tau)\, dx' \\
&\qquad + \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \frac{\partial E}{\partial x_n'}(x, 0; y, \tau)\, dx' \\
&\quad = I_1 + I_2.
\end{aligned}$$

By Lemma 2.4.12, $I_2 \to 0$ as $(y, \tau)$ $(\in \overline{\Gamma}_\beta(0)) \to (0, 0)$. Also

$$\begin{aligned}\left|I_1 - \frac{\beta}{\sqrt{4\pi}}\right| &\le \left|I_1 - \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \sum a_{ij}(y')n_i(x)\frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\,dx'\right| \\ &\quad + \left|\int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \sum a_{ij}(y')n_i(x)\frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\,dx' - \frac{\beta}{\sqrt{4\pi}}\right| \\ &= E_1 + E_2.\end{aligned}$$

By Lemma 2.4.15, $E_2 \to 0$ as $(y, \tau)$ $(\in \overline{\Gamma}_\beta(0)) \to (0, 0)$. While

$$\begin{aligned} E_1 &= \left|I_1 - \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \sum a_{nj}(y')\frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\,dx'\right| \quad (n_i(x))_{i=1}^n = (0, \dots, 0, 1) \\ &\le C\left\{\sum_{1\le j\le n-1} |a_{nj}(y')| + |a_{nn}(y') - 1|\right\}\int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \max_j \left|\frac{\partial G_y}{\partial x_j}(x, 0; y, \tau)\right| dx' \\ &\le C\left\{\sum_{1\le j\le n-1} \left|\frac{\partial \phi}{\partial y_j}(y')\right| + |\nabla_{y'}\phi(y')|^2\right\} \\ &\quad \cdot \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \frac{\delta(y)}{|\tau|^{n+2/2}} \exp\left(-C\frac{|x-y|^2}{|\tau|}\right) dx' \\ &\le C'\left\{\sum_{1\le j\le n-1} \left|\frac{\partial \phi}{\partial y_j}(y')\right| + |\nabla_{y'}\phi(y')|^2\right\} \\ &\quad \cdot \frac{1}{|\tau|^{n-1/2}} \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \exp\left(-C'\frac{|x'|^2}{|\tau|}\right) dx' \\ &\le C'\left\{\sum_{1\le j\le n-1} \left|\frac{\partial \phi}{\partial y_j}(y')\right| + |\nabla_{y'}\phi(y')|^2\right\} \to 0 \end{aligned}$$

as $(y, \tau)$ $(\in \overline{\Gamma}_\beta(0)) \to (0, 0)$ since $\nabla\phi(0) = 0$. Hence $|I_1 - \beta/\sqrt{4\pi}| \to 0$ as $(y, \tau)$ $(\in \overline{\Gamma}_\beta(0)) \to (0, 0)$. So

$$\lim_{\substack{(y,\tau)\in\overline{\Gamma}_\beta(0) \\ \tau\to 0}} \int_{\substack{|x'|\le\varepsilon \\ x'\in R^{n-1}}} \frac{\partial \widetilde{G}}{\partial x_n'}(x, 0; y, \tau)\,dx' = \frac{\beta}{\sqrt{4\pi}}.$$

We are now ready for the proof of Theorem 2.4.1.

*Proof of Theorem* 2.4.1.

$$\begin{aligned}
&\left|\int_{x\in\partial D}\frac{\partial G}{\partial N_x}(x,0;y,\tau)\,d\sigma(x)-\frac{\beta}{\sqrt{4\pi}}\right| \\
&\quad\le\int_{\substack{x\in\partial D\\ |x'|\ge\varepsilon}}\left|\frac{\partial G}{\partial N_x}(x,0;y,\tau)\right|d\sigma(x) \\
&\qquad+\left|\int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}}\nabla_x G(x,0;y,\tau)\cdot(-\nabla\phi(x'),1)\,dx'-\frac{\beta}{\sqrt{4\pi}}\right| \\
&\quad\le\int_{\substack{x\in\partial D\\ |x'|\ge\varepsilon}}\left|\frac{\partial G}{\partial N_x}(x,0;y,\tau)\right|d\sigma(x) \\
&\qquad+\sup_{|x'|\le\varepsilon}|\nabla\phi(x')|\cdot\int_{\substack{x\in\partial D\\ |x'|\le\varepsilon}}|\nabla_x G(x,0;y,\tau)|\,dx' \\
&\qquad+\left|\int_{\substack{|x'|\le\varepsilon\\ x'\in R^{n-1}}}\frac{\partial\widetilde{G}}{\partial x_n'}(x,0;y,\tau)\,dx'-\frac{\beta}{\sqrt{4\pi}}\right| \\
&\quad=I_1'+I_2'+I_3'
\end{aligned}$$

since

$$\frac{\partial G}{\partial x_n}(x,0;y,\tau)=\frac{\partial\widetilde{G}}{\partial x_n'}(x,0;y,\tau)\quad\text{and}\quad\left|\frac{\partial(x',x_n)}{\partial(x',x_n')}\right|=1$$

with $x_n'=x_n-\phi(x')$. Now

$$\begin{aligned}
\int_{\substack{x\in\partial D'\\ |x'|\le\varepsilon}}|\nabla_x G(x,0;y,\tau)|\,dx' &\le\int_{\substack{x\in\partial D'\\ |x'|\le\varepsilon}}\frac{\delta(y)}{|\tau|^{n+2/2}}\exp\left(-C\frac{|x-y|^2}{|\tau|}\right)dx' \\
&\le\int_{\substack{x\in\partial D'\\ |x'|\le\varepsilon}}\frac{1}{|\tau|^{n-1/2}}\exp\left(-C\frac{|x'|^2}{|\tau|}\right)dx'\le C'
\end{aligned}$$

for $(y,\tau)\in\overline{\Gamma}_\beta(0)$.

Therefore $|I_2'|\le C''\sup_{|x'|\le\varepsilon}|\nabla^2\phi(x')|\,|x'|\le C'''\varepsilon$.

By Lemma 2.4.16, $I_3'\to 0$ as $(y,\tau)\in\overline{\Gamma}_\beta(0)\to 0$. Also $I_1'\to 0$ as $\tau\to 0$. Therefore

$$\begin{aligned}
&\varlimsup_{\substack{(y,\tau)\in\overline{\Gamma}_\beta(0)\\ \tau\to 0}}\left|\int_{x\in\partial D}\frac{\partial G}{\partial N_x}(x,0;y,\tau)\,d\sigma(x)-\frac{\beta}{\sqrt{4\pi}}\right|\le C\varepsilon\quad\forall\varepsilon>0 \\
\Rightarrow\;&\lim_{\substack{(y,\tau)\in\overline{\Gamma}_\beta(0)\\ \tau\to 0}}\int_{x\in\partial D}\frac{\partial G}{\partial N_x}(x,0;y,\tau)\,d\sigma(x)=\frac{\beta}{\sqrt{4\pi}}.
\end{aligned}$$

**2.5. An example.** In this section we will give an example of a solution of the heat equation in $R^2$ to show that the index $3/2$ on $t$ in the definition of the space-time cone for the corner points of a cylinder is essentially sharp.

By the result of P. Hartman and A. Wintner [HW] and Hattemer [H], if

$$\theta(x, t) = \sum_{k=-\infty}^{\infty} (4\pi t)^{-1/2} \exp\left(-\frac{(x+2k)^2}{4t}\right),$$

then the Green function for the heat equation in the region $D_1 \times (0, \infty)$, where $D_1 = [0, 1]^2$, is given by

$$G((x_1, x_2), t; (y_1, y_2), \tau) = \prod_{i=1}^{2} \{\theta(x_i - y_i, t-\tau) - \theta(x_i + y_i, t-\tau)\}$$

where $0 \le s < t < \infty$. Therefore

$$\left.\frac{\partial G}{\partial N_{(y_1, y_2)}}\right|_{y_1=0} = -2\theta_{x_1}(x_1, t-\tau) \cdot \{\theta(x_2 - y_2, t-\tau) - \theta(x_2 + y_2, t-\tau)\}$$

where $\partial/\partial N_{(y_1, y_2)}$ is the derivative in the direction of the inward normal $N_{(y_1, y_2)}$ to $\partial D_1$ at $(y_1, y_2) \in \partial D_1$. By the representation theorem in subsection 2.2,

$$u(x, t) = -2\theta_{x_1}(x_1, t-\tau) \cdot \int_0^1 \{\theta(x_2 - y_2, t-\tau) - \theta(x_2 + y_2, t-\tau)\}\, dy_2$$

with $x = (x_1, x_2) \in D_1$, $t > 0$, is the strong solution of the (IDP) in $D_1 \times (0, \infty)$ with initial trace 0 on $D_1$ and trace $d\lambda$ on $\partial D_1$ that equals $dx_2$ on $\{0\} \times [0, 1]$ and equals 0 on $\partial D_1 \backslash \{0\} \times [0, 1]$.

Fix a point $(x_1^0, x_2^0) = (0, x_2^0) \in \partial D$, $0 < x_2^0 < 1$, and let $(x, t) \in D_1 \times (0, \infty)$. Then

$$\int_0^1 \{\theta(x_2 - y_2, t-\tau) - \theta(x_2 + y_2, t-\tau)\}\, dy_2$$

is the bounded solution of the heat equation in $[0, 1] \times (0, \infty)$ with initial value 1 on $(0, 1) \times \{0\}$ and boundary value 0 on the lateral sides of $[0, 1] \times (0, \infty)$. Therefore

$$\int_0^1 \{\theta(x_2 - y_2, t-\tau) - \theta(x_2 + y_2, t-\tau)\}\, dy_2 \to 1 \quad \text{if } |x_2 - x_2^0| \le Ct^{3/2} \to 0.$$

On the other hand,

$$-2\theta_{x_1}(x_1, t-\tau) = 2\sum_{k=-\infty}^{\infty} (4\pi t)^{-1/2} \cdot \frac{2(x_1+2k)}{4t} \exp\left(-\frac{(x_1+2k)^2}{4t}\right)$$
$$= \frac{x_1}{\sqrt{4\pi} t^{3/2}} \exp\left(-\frac{x_1^2}{4t}\right) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(x_1+2k)}{\sqrt{4\pi} t^{3/2}} \exp\left(-\frac{(x_1+2k)^2}{4t}\right).$$

If $x_1 = \beta t^a$, then the second term always goes to 0 as $t \to 0$ while the first term will converge to 0 if $a > 3/2$, $\infty$ if $a < 3/2$, $\beta/\sqrt{4\pi}$ if $a = 3/2$. Hence

$$\begin{aligned} u(x, t) &\to 0 \quad \text{as } t \to 0 \text{ if } a > 3/2 \text{ and } x_1 = \beta t^a,\ |x_2 - x_2^0| \le Ct^{3/2}, \\ &\to \infty \quad \text{as } t \to 0 \text{ if } a < 3/2 \text{ and } x_1 = \beta t^a,\ |x_2 - x_2^0| \le Ct^{3/2}, \\ &\to \frac{\beta}{\sqrt{4\pi}} \quad \text{as } t \to 0 \text{ if } a = 3/2 \text{ and } x_1 = \beta t^a,\ |x_2 - x_2^0| \le Ct^{3/2}. \end{aligned}$$

Therefore the index $3/2$ on $t$ in the definition of the space-time cone is essentially sharp.

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